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RIGHT AND LEFT INVERTIBILITY IN λ - β -CALCULUS (*)

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Abstract. — *A characterization of λ -terms having left and/or right inverses in λ - β -calculus is given and the sets of all and only λ -terms left/right invertible are constructed. The above results are obtained using the concept of Böhm tree, so this study is further used to characterize the λ -terms left/right invertible in the graph model \mathbf{P}_ω .*

Résumé. — *Dans ce papier on va caractériser les λ -termes invertibles à droite et/ou à gauche, en donnant les règles pour construire les deux ensembles constitués respectivement par tous les λ -termes ayant un inverse droite ou gauche. Puisque ces résultats ont été obtenus par la notion d'arbre de Böhm on peut utiliser cette étude au fin de caractériser les λ -termes invertibles à droite ou à gauche dans le modèle \mathbf{P}_ω .*

0. INTRODUCTION

Aim of this paper is the characterization of λ -terms having left and/or right inverses in λ - β -calculus. The semigroup Λ of λ - β -(η)-terms, having the combinator $\mathbf{I} \equiv \lambda x.x$ as identity element and the operation \bullet defined by $X \bullet Y = \mathbf{B}XY$ (where $\mathbf{B} \equiv \lambda xyz.x(yz)$) as composition, has been studied with respect to the left/right invertibility problem in [2], [4], [7, p. 167-168], [8], [9].

In particular in the λ - β -calculus the set of normal forms having at least one left or right inverse has been characterized in [4]. The same paper shows that the combinator \mathbf{I} is the only normal form having both left and right inverse.

The present paper tries to give the final solution to the invertibility problem in λ - β -calculus showing the necessary and sufficient conditions under which an arbitrary λ -term possesses a left (right) inverse and characterizing the set of terms for which there exists only one left (right) inverse; for the

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other left (right) invertible terms an infinite number of inverses is proved to exist. The basic definitions which the paper relies on are those of direct approximation [11], of Böhm tree [1] and of partial order relation \sqsubseteq on the set of λ - $\underline{\Omega}$ -terms, as stated in [10]. Using these notions it is possible to carry on Λ the relation \sqsubseteq defining a λ -term X less or equal to a λ -term Y ($X \sqsubseteq Y$) if and only if its direct approximation $\Phi(X)$ is less or equal to the direct approximation $\Phi(Y)$ of Y ($\Phi(X) \sqsubseteq \Phi(Y)$) and to associate with a λ -term X the approximation set as the set of λ - $\underline{\Omega}$ -terms $\Phi(X')$ such that X is β -convertible to X' . Firstly we notice that every left (right) inverse of a λ -term X is a left (right) inverse of all λ -terms Y such that $X \sqsubseteq Y$. Then in order to characterize the set of terms having left inverse, an operation, called terminal extension, is introduced on the set of Böhm trees. Roughly speaking a terminal extension of a Böhm tree A is a Böhm tree A' obtained from A modifying a terminal node of A either introducing in its label the abstraction of a new variable or pushing the head variable down of a level and substituting it by a bound variable. So we can prove that a λ -term X has a left inverse if and only if there exists in the approximation set of X at least one approximation which can be obtained from **I** applying a sequence of terminal extensions. Moreover it results that every term left invertible, different from **I**, possesses an infinite number of non-convertible left inverses.

The problem of the right invertibility is approached in a similar way. The operation of adding a son with label $\underline{\Omega}$ to the root of a Böhm tree A to obtain a Böhm tree A' is called initial extension. This allows to assert that a λ -term X has right inverse iff there exists at least one approximation of X which can be obtained from **I** applying a sequence of initial extensions. Obviously, as a corollary, it results that **I** is the only λ -term both left and right invertible. Furthermore we can prove that the number of right inverses for a right invertible term X is either one or infinite depending on the form of the term itself.

Finally we notice that the above results about invertibility can be carried on the graph model \mathbf{P}_ω [1, p. 467] and we show that the two functions which map an element of \mathbf{P}_ω into the set of all its right or left inverses, respectively, are not monotonic, i. e. it is possible to find a left (right) inverse of an element X of \mathbf{P}_ω which is not a left (right) inverse of an element Y , whereas $X \sqsubseteq Y$ (\sqsubseteq is the usual order relation on \mathbf{P}_ω).

1. NOTATIONS AND DEFINITIONS

In the sequel we will use the following notions and conventions:

- i) λ -calculus means λ - β -calculus, normal form λ - β -normal form, \geq , $=$, \equiv

denote β -reducibility, α - β -convertibility and modulo α identity, respectively; moreover Λ represents the set of λ -terms;

ii) the word combinator will refer to closed λ -terms, i. e. terms without free variables; the combinators will be indicated by uppercase, boldface characters, for example $\mathbf{B} \equiv \lambda x y z . x(yz)$, $\mathbf{I} \equiv \lambda x . x$, etc.;

iii) we indicate by means of the ordered sequences of λ -terms

$$\langle X_0, X_1, \dots, X_k \rangle$$

the λ -terms $\lambda z . z X_0 X_1 \dots X_k$ where z does not occur free in any X_i , $0 \leq i \leq k$ (Church n -tuple) [6];

iv) $C[]$ denotes a context, i. e. a λ -term where one subterm is missing; $C[X]$ then denotes the result of filling the missing subterm with X (for a more formal definition see [11]);

v) $X[x := Y]$ indicates the λ -term obtained from a λ -term X by substituting in it the λ -term Y to every free occurrence (if any) of the variable x .

As the concept of approximation of a λ -term [11] and the related one of Böhm tree [1, p. 211] are very useful for this study, we summarize here the principal definitions and conventions about them.

A λ -term X has *head normal form* if it has the form $\lambda x_1 x_2 \dots x_m . y X_1 X_2 \dots X_n$ where:

- x_1, x_2, \dots, x_m are variables and $m \geq 0$;
- X_1, X_2, \dots, X_n are λ -terms and $n \geq 0$;
- y is a variable, free or bound (as usual it will be called the head variable of X).

The *direct approximation* $\Phi(X)$ of a λ -term X is defined as follows:

$$\Phi(X) = \lambda x_1 \dots x_m . y \Phi(X_1) \Phi(X_2) \dots \Phi(X_n) \text{ if } X = \lambda x_1 \dots x_m . y X_1 X_2 \dots X_n;$$

$$\Phi(X) = \underline{\Omega}, \text{ where } \underline{\Omega} \text{ is an extra constant, if } X \text{ has not a head normal form.}$$

The set $\Phi(\Lambda)$ will be indicated by \mathcal{N} (set of λ - $\underline{\Omega}$ -terms). Inside \mathcal{N} the following partial order relation \sqsubseteq is defined [10]: for any M, N of \mathcal{N} $M \sqsubseteq N$ iff either

- i) $M \equiv \underline{\Omega}$; or
- ii) $M \equiv \lambda x_1 x_2 \dots x_n . x_j M_0 \dots M_k$
 $N \equiv \lambda x_1 x_2 \dots x_n . x_j N_0 \dots N_k$

and $M_i \sqsubseteq N_i$ for any i ($0 \leq i \leq k$).

Given a λ -term X we call *approximation set* of X : $\mathcal{A}(X)$ the subset of \mathcal{N} so defined:

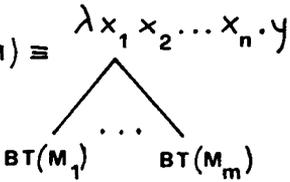
$$\mathcal{A}(X) = \{ M \in \mathcal{N} \mid M \sqsubseteq \Phi(X) \}.$$

The partial order relation \sqsubseteq can be carried on Λ as follows: for any X, Y of $\Lambda, X \sqsubseteq Y$ iff $\Phi(X) \sqsubseteq \Phi(Y)$.

We can visualize every element M of \mathcal{N} by means of a suitable tree: the Böhm tree (B. T.) of M . Given an element M of \mathcal{N} , the B. T. of M : $BT(M)$ is the labelled tree so defined:

i) if $M \equiv \underline{\Omega}$ $BT(M) \equiv \bullet \underline{\Omega}$

ii) if $M \equiv \lambda x_1 x_2 \dots x_n . y M_1 \dots M_m$ $BT(M) \equiv$



We will refer to \mathcal{B} as to the set of the B. T. of the elements of \mathcal{N} . The nodes of a B. T. will be indicated by strings of natural numbers (included the empty string ϵ , labelling the root) in the usual way, so that β denotes a successor of α if and only if α is a prefix of β : $\beta = \alpha\gamma$ for some γ . Let A be a B. T. and α be a node with label $\lambda x_1 \dots x_n . y$, in the sequel we will use the following conventions [see 1, p. 218]:

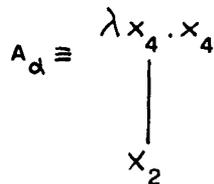
- i) A_α indicates the subtree of A having as root the node α ;
- ii) $\bar{\alpha}$ indicates the path from the root to the node α ;
- iii) $b(\alpha)$ indicates the vector of the bound variables occurring in the label of α , i. e. $b(\alpha) = x_1 x_2 \dots x_n$;
- iv) $b(\bar{\alpha})$ indicates the vector of the bound variables occurring in the labels of the nodes of the path $\bar{\alpha}$, inductively defined as follows:

- $b(\bar{\epsilon}) = b(\epsilon)$
- $b(\bar{\alpha'} \langle k \rangle) = b(\alpha') b(k)$.

By way of example, for the B. T. A of figure 1, if we choose as node α the node $\langle 10 \rangle$, we have:

$$b(\alpha) = x_4$$

$$b(\bar{\alpha}) = x_0 x_1 x_2 x_3 x_4$$



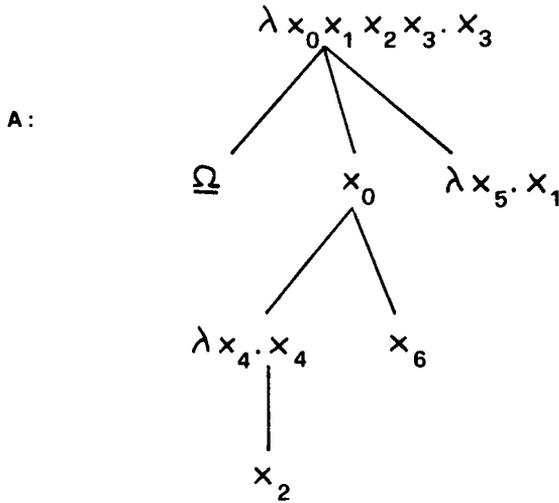


Figure 1. — A Böhm tree A.

By stretching the Böhm tree definition, in the sequel sometimes we will refer to the B. T. of an element X of Λ : $BT(X)$, as to the B. T. of its direct approximation.

Obviously any B. T. A of \mathcal{B} will define one and only one term of \mathcal{N} : M_A such that $BT(M_A) = A$ (for example for the B. T. A of figure 1

$$M_A \equiv \lambda x_0 x_1 x_2 x_3 . x_3 \underline{\Omega}(x_0(\lambda x_4 . x_4 x_2)x_6)\lambda x_5 . x_1);$$

hence the order relation \sqsubseteq on \mathcal{N} can be carried on \mathcal{B} : $A \sqsubseteq B$ iff $M_A \sqsubseteq M_B$.

2. RIGHT AND LEFT INVERTIBILITY

Aim of this section is to study the conditions under which an arbitrary λ -term X has right and/or left inverses. In the sequel we use the following notations:

i) $X_R(X_L)$ denotes a right (left) inverse of a λ -term X , i. e.,:

$$\mathbf{B}X X_R = \mathbf{I} \quad (\mathbf{B}X_L X = \mathbf{I}).$$

ii) $\mathcal{I}_R(X)$ ($\mathcal{I}_L(X)$) denotes the set of all the right (left) inverses of a λ -term X .

THEOREM 1: Let X, Y be two λ -terms of Λ for which $X \sqsubseteq Y$, then:

- i) $\mathcal{I}_R(X) \subseteq \mathcal{I}_R(Y)$
- ii) $\mathcal{I}_L(X) \subseteq \mathcal{I}_L(Y)$.

Proof: i) The assertion is trivially true for $\mathcal{S}_R(X)$ empty.

If it is not true, we prove that any right inverse X_R of X is also a right inverse for Y . By definition we have:

$$X(X_R y) \geq y \quad \text{for any variable } y \text{ not free in } X \text{ and } X_R.$$

Since Lévy has proved (th. 5. 8, p. 105 of [10]) that if $X \sqsubseteq Y$ then $C[X] \sqsubseteq C[Y]$ for any context $C[]$, if we choose as context $[](X_R y)$ it will be:

$$y \leq X(X_R y) \sqsubseteq Y(X_R y) \quad \text{hence} \quad Y(X_R y) \geq y.$$

ii) The proof is analogous to the preceding one if we choose as context $X_L([]y)$.

2.1. Left Invertibility

DEFINITION 2.1.1: Let A, A' be two Böhm trees and α a terminal node of A with label $\lambda x_{i_1} x_{i_2} \dots x_{i_h} . x_t$. We say that A' is a *terminal extension of A in α* if A' results from A by one of the following substitutions:

- 1) the label of the node α in A is replaced in A' by the label

$$\lambda x_{i_1} x_{i_2} \dots x_{i_h} x_{i_{h+1}} . x_t \quad (\text{terminal extension of type 1});$$

- 2) the subtree A_α is replaced in A' by a subtree A'_α such that:

a) the label of α is $\lambda x_{i_1} \dots x_{i_h} . x_j$, where x_j is a bound variable distinct from x_t ;

b) α has m sons with $m \geq 1$. Each of these sons are terminal nodes, one and only one of them has label x_t , whereas the remaining $m - 1$ have label $\underline{\Omega}$ (see fig. 2) (terminal extension of type 2).

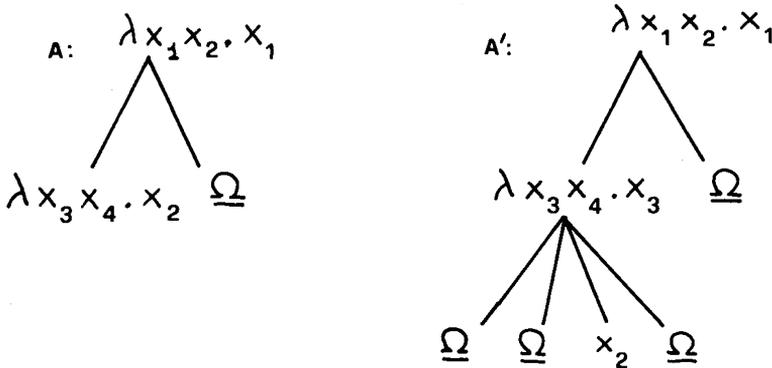


Figure 2. — A terminal extension of type 2.

With every terminal extension e of type 2, we associate the triple

$$\tau(e) = \langle x_j, m, k \rangle,$$

where x_j and m are respectively the name of the head variable and the number of sons of the node α in A' and k indicates that the only son of α in A' with label different from Ω is the k -th.

DEFINITION 2.1.2: Let A, A' be two Böhm trees.

We say that A' is a terminal extension of A ($A \xrightarrow{t\text{-ext}} A'$) if it is a terminal extension of A in some terminal node.

DEFINITION 2.1.3: We call *Left Invertible Term Generator Set* the subset $\mathcal{L} \subset \mathcal{N}$ inductively defined as follows:

- i) $\mathbf{I} \in \mathcal{L}$
- ii) $N \in \mathcal{L}$ and $BT(N) \xrightarrow{t\text{-ext}} BT(N') \Rightarrow N' \in \mathcal{L}$.

DEFINITION 2.1.4: Let N be an element of \mathcal{L} . We call *history* of N : $\mathcal{H}(N)$ a sequence of elements of \mathcal{L} : $\langle N^0, N^1, \dots, N^h \rangle$ such that $N^0 \equiv \mathbf{I}$, $N^h \equiv N$ and for any i , $0 \leq i \leq h-1$, $BT(N^i) \xrightarrow{t\text{-ext}} BT(N^{i+1})$.

LEMMA 2.1.1: Every element N of \mathcal{L} has one and only one history: $\mathcal{H}(N)$.

Proof: Obvious from definition 2.1.1 and definition 2.1.3.

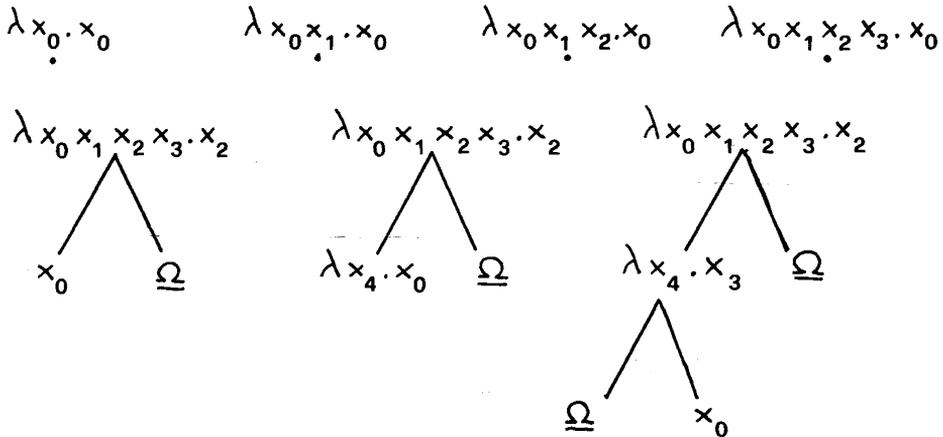


Figure 3. — Böhm trees of the history of the λ - Ω -term $\lambda x_0 x_1 x_2 x_3 . x_2 (\lambda x_4 . x_3 \Omega x_0) \Omega$.

DEFINITION 2.1.5: Let N be an element of \mathcal{L} . We say that N is a *term non-homogeneous for the variable x_i* if in its history $\mathcal{H}(N)$ there are at least two

terminal extensions e, e' of type 2 with $\tau(e) = \langle x_i, m, k \rangle$ and $\tau(e') = \langle x_i, m', k' \rangle$ such that $m \neq m'$ and/or $k \neq k'$.

Figure 4(a) shows the Böhm tree of a term non-homogeneous for the variable x_1 , whereas it is homogeneous for the variable x_2 ; instead the term whose Böhm tree is in figure 4(b) is homogeneous for each variable occurring as head variable; in such a case we say that the term is *homogeneous*.

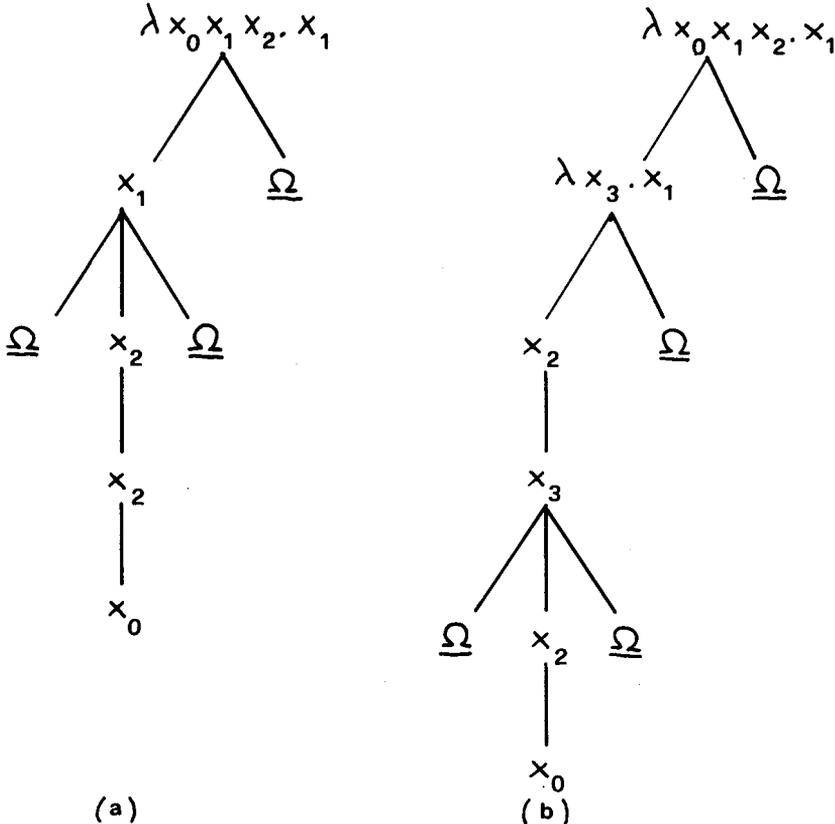


Figure 4. — Böhm trees of a non-homogeneous (a) and of a homogeneous λ - $\underline{\Omega}$ -term (b).

From lemma 3 of [3] it follows lemma 2.1.2 which has been rewritten and proved (in a simpler way) using the notation of the present work.

LEMMA 2.1.2: Let N be a λ - $\underline{\Omega}$ term of \mathcal{L} , non-homogeneous for a set of variables $\{x_{l_1}, x_{l_2}, \dots, x_{l_k}\}$. We state that there is a normal combinator

$C_{[m]}I$ such that the term $N' [x_{l_i} := C_{[m]}I]$, where N' is obtained from N by eliminating the abstraction of x_{l_i} , is non-homogeneous for the set

$$\{ x_{l_1}, x_{l_2}, \dots, x_{l_{i-1}}, x_{l_{i+1}}, \dots, x_{l_k} \}.$$

Proof: Let e_1, e_2, \dots, e_n be the terminal extensions of type 2, occurring in $\mathcal{H}(N)$ such that the first element of $\tau(e_j)$ is $l_i (1 \leq j \leq n)$, i. e.:

$$\begin{aligned} \tau(e_1) &= \langle x_{l_i}, m_1, k_1 \rangle \\ \tau(e_2) &= \langle x_{l_i}, m_2, k_2 \rangle \\ &\dots \\ \tau(e_n) &= \langle x_{l_i}, m_n, k_n \rangle. \end{aligned}$$

Let $m = \max(m_1, m_2, \dots, m_n)$. It is easy to prove that the normal combinator $C_{[m]}I \equiv \lambda t_0 t_1 \dots t_m . t_m t_0 t_1 \dots t_{m-1}$ satisfies the thesis, because it substitutes the different occurrences of x_{l_i} by different variables.

LEMMA 2.1.3: Every λ -term X of Λ , whose direct approximation is in \mathcal{L} , has at least a left inverse.

Proof: Firstly we prove that every λ -term X , whose direct approximation is a homogeneous element of \mathcal{L} has a left inverse. From definition 2.1.3 it follows that there is one and only one terminal node of $BT(X)$ having label different from $\underline{\Omega}$; let such a node be α and let $b(\bar{\alpha}) = x_0 x_1 \dots x_n, n \geq 0$. We assert that there are n suitable λ -terms $\Psi_1, \Psi_2, \dots, \Psi_n$ such that the sequence $\langle \Psi_1, \Psi_2, \dots, \Psi_n \rangle$ is a left inverse for X . We prove this assertion by induction on the number h of elements of $\mathcal{H}(\Phi(X))$.

$$\begin{array}{lll} h=1. & X = I & X_L = \lambda z . z \equiv I. \\ h+1. & & \end{array}$$

Given $\mathcal{H}(\Phi(X)) = \langle N^0, N^1, \dots, N^h, N^{h+1} \rangle$, let $X^i, 0 \leq i \leq h$, be a λ -term such that $\Phi(X^i) = N^i$, let $X^{h+1} = X$ and $A^i = BT(N^i)$. We distinguish two cases either A^{h+1} extends A^h by a terminal extension of type 1 or A^{h+1} extends A^h by a terminal extension of type 2. In the first case we say that a left inverse for X can be obtained by adding to the left inverse of X^h (existing by induction hypothesis) a generic λ -term Ψ_n , i. e.

$$\begin{array}{l} \text{if} \\ X_L^h = \langle \Psi_1^h, \Psi_2^h, \dots, \Psi_{n-1}^h \rangle \\ X_L^{h+1} \text{ will be } \langle \Psi_1^h, \Psi_2^h, \dots, \Psi_{n-1}^h, \Psi_n \rangle. \end{array}$$

In fact it follows from the definitions of \mathcal{L} and of terminal extension of type 1 that:

$$(X^{h+1}y) = (X^h y)[y := \lambda x_n . y]$$

and by induction hypothesis:

$$\begin{aligned} \text{i. e.:} \quad & (X^{h+1}y)\Psi_1^h\Psi_2^h \dots \Psi_{n-1}^h \geq \lambda x_n \cdot y \\ & (\lambda x_n \cdot y)\Psi_n \geq y. \end{aligned}$$

In the second case, let $\langle x_j, m, k \rangle$ be the triple associated with the $(h+1)$ -th terminal extension. If x_j occurs as head variable in some terminal extension preceding the $(h+1)$ -th one, from homogeneity hypothesis it follows that the left inverse X_L^h (existing by induction hypothesis) is also a left inverse for X^{h+1} ; otherwise we prove that a left inverse of X^{h+1} can be obtained by substituting in the left inverse X_L^h for the λ -term Ψ_j^h the normal combinator (selector)

$$\begin{aligned} \mathbf{U}_k^m &= \lambda t_1 t_2 \dots t_m \cdot t_k, \quad \text{i. e.:} \\ X_L^{h+1} &= \langle \Psi_1^h, \Psi_2^h, \dots, \Psi_{j-1}^h, \mathbf{U}_k^m, \Psi_{j+1}^h, \dots \rangle. \end{aligned}$$

It follows from definitions of \mathcal{L} and of terminal extension of type 2, that:

$$(X^{h+1}y) = (X^h y)[y := x_j X'_1 X'_2 \dots X'_{k-1} y X'_{k+1} \dots X'_m]$$

where X'_i are unsolvable terms; then:

$$(X^{h+1}y)\Psi_1^h\Psi_2^h \dots \Psi_{j-1}^h \mathbf{U}_k^m \Psi_{j+1}^h \dots \geq \mathbf{U}_k^m X'_1 X'_2 \dots X'_{k-1} y X'_{k+1} \dots X'_m \geq y.$$

Now, let us suppose that X has a direct approximation non-homogeneous only for one variable x_i . From lemma 2.1.2 it follows that there exists an integer m such that the term $N' [x_i := \mathbf{C}_{[m]} \mathbf{I}]$, where N' is obtained from $\Phi(X)$ by eliminating the abstraction of x_i , is homogeneous. Let X' be a λ -term of Λ such that $\Phi(X') = N' [x_i := \mathbf{C}_{[m]} \mathbf{I}]$ and let X'_L be its left inverse, existing for the first part of this lemma: $X'_L = \langle \Psi'_1, \Psi'_2, \dots, \Psi'_n \rangle$. We maintain that the sequence $X_L = \langle \Psi'_1, \Psi'_2, \dots, \Psi'_{i-1}, \mathbf{C}_{[m]} \mathbf{I}, \Psi'_i, \Psi'_{i+1}, \dots, \Psi'_n \rangle$ is a left inverse for X . In fact:

$$\begin{aligned} (X'y)\Psi'_1\Psi'_2 \dots \Psi'_{i-1} &= (Xy)\Psi'_1\Psi'_2 \dots \Psi'_{i-1}(\mathbf{C}_{[m]} \mathbf{I}) \\ (Xy)\Psi'_1\Psi'_2 \dots \Psi'_{i-1}(\mathbf{C}_{[m]} \mathbf{I})\Psi'_i \dots \Psi'_n &= (X'y)\Psi'_1\Psi'_2 \dots \Psi'_{i-1}\Psi'_i \dots \Psi'_n \geq y. \end{aligned}$$

The proof can be generalized in an obvious way to the case of terms non-homogeneous for more than one variable.

LEMMA 2.1.4: Every λ -term of Λ , distinct from \mathbf{I} and having the direct approximation in \mathcal{L} , has an infinite number of non convertible left inverses.

Proof: Let X be a λ -term satisfying the hypothesis of this lemma. If some of the λ -terms of the not empty sequence X_L , obtained by the construction of lemma 2.1.3, are arbitrary we can obtain an infinite number of left inverses choosing them in infinite ways.

Instead if each Ψ_i has been substituted by a suitable combinator, we can obtain an infinite number of left inverses as follows. Let U_k^m be a selector occurring in X_L (from proof of lemma 2.1.3 it is clear that in X_L we have surely some selectors), i. e.:

$$X_L = \langle \Psi_1, \Psi_2, \dots, \Psi_{i-1}, U_k^m, \Psi_{i+1}, \dots, \Psi_k \rangle.$$

It is easy to prove that

$$X'_L = \langle \Psi_1, \Psi_2, \dots, \Psi_{i-1}, U_k^{m+n}, \Psi_{i+1}, \dots, \Psi_h, \Phi_1, \dots, \Phi_n \rangle$$

where $\Phi_i (1 \leq i \leq n)$ are generic λ -terms, is another left inverse for X , non-convertible to X_L :

$$\begin{aligned} X'_L(Xy) &\geq (Xy)\Psi_1\Psi_2 \dots \Psi_{i-1}U_k^{m+n}\Psi_{i+1} \dots \Psi_n\Phi_1\Phi_2 \dots \Phi_n \geq \\ &\geq (\lambda t_1 t_2 \dots t_n. y)\Phi_1\Phi_2 \dots \Phi_n \geq y. \end{aligned}$$

DEFINITION 2.1.6: A λ -term X of Λ is of type Σ if the set $\mathcal{A}(X) \cap \mathcal{L}$ is not empty.

Remark 1: For any Böhm tree $BT(X)$ of a λ -term X of type Σ (shortly B. T. of type Σ), there is at least a terminal node σ , such that:

- i) the first component of the vector $b(\bar{\sigma})$ occurs as head variable only in the label of σ ;
- ii) every head variable in the label of a not terminal node of the path $\bar{\sigma}$, is bound.

The Böhm tree of figure 5 is of type Σ , because the terminal nodes $\langle 2 \rangle$ and $\langle 11 \rangle$ satisfy the conditions of the remark 1.

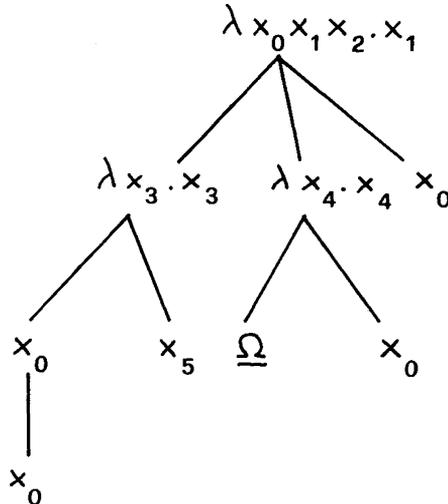


Figure 5. — A Böhm tree of type Σ .

THEOREM 2.1.1: A λ -term X has at least a left inverse if and only if it is of type Σ .

Proof: If X is of type Σ , there is at least an approximation $N' \sqsubseteq \Phi(X)$ belonging to \mathcal{L} , so for theorem 2.1 and lemma 2.1.3 X has at least a left inverse.

Now, let us suppose, *per absurdum*, that the λ -term X not of type Σ has a left inverse. If X is not of type Σ one of the conditions of remark 1 is not satisfied.

If for any path $\bar{\sigma}$ of $BT(X)$ the condition *i*) of remark 1 does not hold, then in (Xy) the free variable y , if it occurs, always occurs applied to a positive number of arguments, which cannot be eliminated using only β -reductions. Instead if for any path for which condition *i*) of remark 1 holds, there is some non-terminal node whose label has as head variable a free variable, then there is no λ -term Y such that in $Y(Xy)$ this free variable can be erased to obtain y .

2.2. Right Invertibility

DEFINITION 2.2.1: Let A, A' be two B. T., different from $\underline{\Omega}$. We say that A' is an *initial extension* of A ($A \xrightarrow{i-ext} A'$) if A' results from A by adding to its root a son with label $\underline{\Omega}$ (see fig. 6)

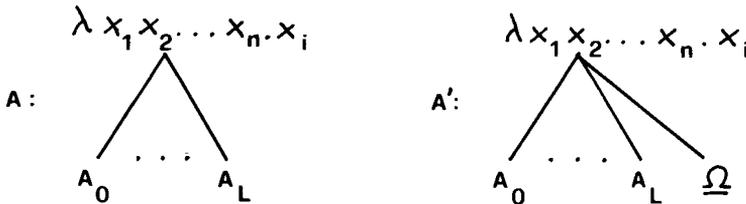


Figure 6. — Two Böhm trees A and A' such that $A \xrightarrow{i-ext} A'$.

DEFINITION 2.2.2: We call *Right Invertible Term Generator Set* the subset $\mathcal{R} \subset \mathcal{N}$ inductively defined as follows:

- i) $I \in \mathcal{R}$
- ii) $N \in \mathcal{R}$ and $BT(N) \xrightarrow{i-ext} BT(N') \Rightarrow N' \in \mathcal{R}$.

LEMMA 2.2.1: Every λ -term X of Λ , whose direct approximation $\Phi(X)$ is in \mathcal{R} has one and only one right inverse X_R .

Proof: “ One ” part. Let X be $\lambda x. xX_1X_2 \dots X_h$, with $X_i(1 \leq i \leq h)$ unsolvable; we take as X_R the λ -term $U_1^{h+1} \equiv \lambda x_0 x_1 \dots x_h. x_0$. It's trivial that $X(X_R y) \geq y$, so X_R is a right inverse for X .

“ Only one ” part. Let us suppose, *per absurdum*, that $X = \lambda z. zX_1 \dots X_h$,

with $X_i (1 \leq i \leq h)$ unsolvable, has a right inverse $X'_R = \lambda x_0 x_1 \dots x_n \cdot x_j Y_1 Y_2 \dots Y_t$ distinct from $X_R = \lambda x_0 x_1 \dots x_h \cdot x_0$.

Since, from the definition of right inverse,

$$(X'_R y) X_1 \dots X_h \geq y$$

we must have $n \leq h$, otherwise we cannot eliminate the $n - h$ initial abstractions.

Since from theorem 2.1.1 it follows that X'_R is of type Σ , the head variable x_j of X'_R must be exactly x_0 if $t=0$, different from x_0 and bound if $t \neq 0$. In the first case we must have $n=h$, otherwise y remains applied to a positive number of λ -terms, which cannot be eliminated to give y , hence $X'_R = X_R$, contrary to the hypothesis. In the second case, we should have, for some X_j unsolvable:

$$X_j Y'_1 \dots Y'_t X_{n+1} \dots X_h \geq y,$$

where:

$$Y'_i = Y_i [x_0 := y, x_1 := X_1, \dots, x_n := X_n] \quad \text{for } 1 \leq i \leq t$$

and this is an *absurdum*.

DEFINITION 2.2.3: We say that a λ -term X of Λ is of type Ξ if the set $\mathcal{A}(X) \cap \mathcal{R}$ is not empty.

Example: The λ -terms whose B. T. is shown in figure 7 are of type Ξ , because they have as approximation the λ - Ω -term $\lambda x_0 \cdot x_0 \underline{\Omega \Omega \Omega}$.

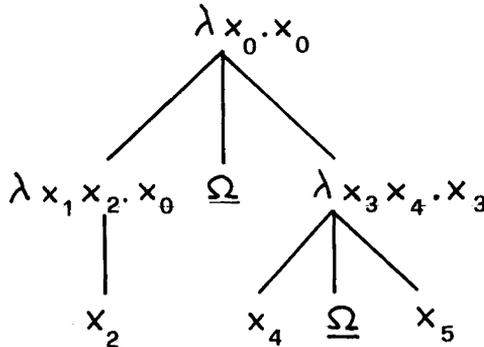


Figure 7. — A Böhm tree of λ -terms of type Ξ .

Remark 2: If X is of type Ξ , it has the form $\lambda x \cdot x X_1 \dots X_h$.

THEOREM 2.2.1: A λ -term X has at least a right inverse if and only if it is of type Ξ .

Proof: If X is a λ -term of type Ξ , $\mathcal{J}_R(X)$ is not empty from lemma 2.2.1 and theorem 2.1. Now let us suppose X not of type Ξ ; then X can have in its head more than one abstraction: $X = \lambda x_0 x_1 \dots x_n . x_i X_1 \dots X_h$ and/or X can have as head variable a free variable $X = \lambda x_0 x_1 \dots x_n . y X_1 \dots X_h$. In the first case the $n+1$ initial abstractions cannot be eliminated using β -reductions; in the second case the free variable cannot be erased.

COROLLARY: The only λ -term having left and right inverse is the combinator **I**.

DEFINITION 2.2.4: We say that a λ -term X is Ω -like if either:

- i) X is unsolvable, or
- ii) X is solvable and its head variable is free.

We say X not Ω -like on the contrary.

LEMMA 2.2.2: Let X be a λ -term:

- i) if X is Ω -like, for any λ -term Y the application $(X Y)$ is also an Ω -like term.
- ii) if X is Ω -like, for any variable y different from the head variable of X , if any, there are no h λ -terms Y_1, Y_2, \dots, Y_h such that:

$$X Y_1 Y_2 \dots Y_h \geq y$$

- iii) if X is not Ω -like there are h λ -terms Y_1, Y_2, \dots, Y_h such that:

$$X Y_1 Y_2 \dots Y_h \geq \mathbf{I}.$$

Proof: Both assertions i) and ii) are trivially true for X unsolvable. Let us suppose X solvable with head variable free: $X = \lambda x_1 x_2 \dots x_k . a X_1 \dots X_h$, then the head variable a cannot be eliminated using only β -reductions, so $(X Y)$ is solvable with head variable a , moreover it is impossible to reduce X to a free variable y different from a .

To prove assertion iii), let us suppose $X = \lambda x_1 x_2 \dots x_k . x_j X_1 \dots X_s$, with x_j bound. If we choose $h = k$, $Y_i = \Psi_i$, where Ψ_i is a generic λ -term, for $1 \leq i < j$ and $j < i \leq h$, and $Y_j = U_{s+1}^{s+1}$, where $U_{s+1}^{s+1} \equiv \lambda x_0 x_1 \dots x_s . x_s$, it is trivially true that $X Y_1 \dots Y_h \geq \mathbf{I}$.

THEOREM 2.2.2: Let X be a λ -term of type Ξ : $X = \lambda z . z X_1 \dots X_h$. If every X_i is Ω -like, then X has one and only one right inverse, else X has an infinite number of right inverses.

Proof: Let us suppose $X = \lambda z . z X_1 \dots X_h$ with $X_i (1 \leq i \leq h)$ Ω -like. We

must prove that X has only the right inverse given in the proof of lemma 2.2.1: $X_R = \lambda x_0 \dots x_h. x_0$.

The existence of another inverse X'_R should cause an *absurdum*, in fact being $X'_R = \lambda x_0 x_1 \dots x_n. x_j Y_1 \dots Y_h$ of type Σ because of theorem 2.1.1, its head variable must be bound and different from x_0 (see proof of lemma 2.2.1), then we should have, for some X_i Ω -like and some Z_1, Z_2, \dots, Z_k :

$$X_i Z_1 \dots Z_k \geq y$$

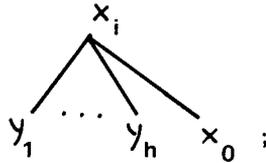
and this is an *absurdum* because of lemma 2.2.2, case ii).

Now let us suppose that at least one λ -term X_i is not Ω -like. For lemma 2.2.2, case iii), there exist h λ -terms Y_1, Y_2, \dots, Y_h such that:

$$X_i Y_1 Y_2 \dots Y_h \geq \mathbf{I}.$$

Let $\mathcal{S}(X)$ be the set inductively defined as follows:

- i) $X_R = \lambda x_0 \dots x_h. x_0$ is in $\mathcal{S}(X)$
- ii) if Y is in $\mathcal{S}(X)$ and Y' is a term obtained by substituting in the $BT(Y)$ to the terminal node x_0 the subtree:



then Y' is in $\mathcal{S}(X)$.

It is obvious that $\mathcal{S}(X)$ has an infinite number of elements, which are all right inverses of X .

3. LEFT AND RIGHT INVERTIBILITY IN THE GRAPH MODEL \mathbf{P}_ω

H. Barendregt [1, p. 496-500], reformulating in terms of Böhm trees the Hyland's characterization of the equality in the graph model \mathbf{P}_ω , has shown that

$$\mathbf{P}_\omega \models X = Y \Leftrightarrow BT(X) = BT(Y).$$

So we can say that the above results about invertibility on \mathcal{N} (or \mathcal{B}) can be carried on \mathbf{P}_ω . Now let f and g be the following functions

$$\begin{aligned} f: \mathbf{P}_\omega &\rightarrow 2^{\mathbf{P}_\omega} & f(X) &= \mathcal{I}_L(X) \\ g: \mathbf{P}_\omega &\rightarrow 2^{\mathbf{P}_\omega} & g(X) &= \mathcal{I}_R(X), \end{aligned}$$

since both \mathbf{P}_ω and $2^{\mathbf{P}_\omega}$ are complete lattices [1, p. 19], it is of some interest to investigate whether f and g are monotonic functions, i. e.

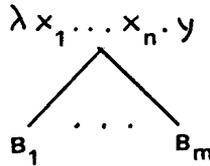
whether $\mathcal{I}_L(X) \subseteq \mathcal{I}_L(Y)$ whenever $X \sqsubseteq Y$
 and whether $\mathcal{I}_R(X) \subseteq \mathcal{I}_R(Y)$ whenever $X \sqsubseteq Y$,

being \sqsubseteq the order relation on \mathbf{P}_ω . H. Barendregt [1, p. 228-240, 496-500] has shown that

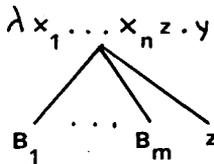
$$\mathbf{P}_\omega \models X \sqsubseteq Y \Leftrightarrow BT(X)\eta_{\sqsubseteq}BT(Y),$$

where η_{\sqsubseteq} is the order relation defined as it follows.

DEFINITION 3.1: Let A be a B. T. and α one of its nodes having label $\lambda x_1 \dots x_n. y$. The B. T. A' is an η -expansion of A at α if it results from A by replacing the subtree A_α , which has the form



by the subtree A'_α having the form



In the sequel if α is the root of A , we call the η -expansion at α *head η -expansion*, if α is a terminal node of A we call it *terminal η -expansion*.

DEFINITION 3.2: Let A, A' be two Böhm trees. A' is a (possibly) infinite η -expansion of A (shortly $A \leq_\eta A'$) if it results from A by the application of a (possibly infinite) sequence of η -expansions.

DEFINITION 3.3: Let A, A' be two Böhm trees. $A \eta_{\sqsubseteq} A'$ if there exists a Böhm tree B , which is a (possibly) infinite η -expansion of A , such that $B \sqsubseteq A'$, i. e. $A \leq_\eta B \sqsubseteq A'$.

In the sequel if $A \eta_{\sqsubseteq} B$, i. e. $A \leq_\eta A' \sqsubseteq B$ for some A' , and no terminal (head) η -expansion is applied to A in order to obtain A' , we say that B results from A without terminal (head) η -expansions.

LEMMA 3.1: Let X, Y be two λ -terms for which $BT(X) \eta_{\sqsubseteq} BT(Y)$ and let X be of type Ξ . Y is of type Ξ if and only if $BT(Y)$ results from $BT(X)$ without head η -expansions.

Proof: Obvious.

THEOREM 3.1: The function g is not monotonic.

Proof: Let X be a λ -term of type Ξ and let Y be a λ -term for which $BT(X) \eta_{\sqsubseteq} BT(Y)$. If $BT(Y)$ results from $BT(X)$ by some head η -expansion, from lemma 3.1 it follows that $\mathcal{I}_R(Y)$ is empty so $\mathcal{I}_R(X) \not\subseteq \mathcal{I}_R(Y)$, being $\mathcal{I}_R(X)$ not empty.

Notice that also in the case in which $BT(Y)$ results from $BT(X)$ without head η -expansions we can have $\mathcal{I}_R(X) \not\subseteq \mathcal{I}_R(Y)$. For example if

$$X = \lambda x_0. x_0(\lambda x_1. x_1) \quad \text{and} \quad Y = \lambda x_0. x_0(\lambda x_1 x_2. x_1 x_2),$$

we have that $X_R = \lambda t_0 t_1. t_1 t_0$ is a right inverse for X but not for Y .

LEMMA 3.2: Let X, Y be two λ -terms for which $BT(X) \leq_{\eta} BT(Y)$ and let X be of type Σ . Y is of type Σ if and only if there exists $A \in \mathcal{A}(X) \cap \mathcal{L}$ such that $BT(Y)$ results from $BT(A)$ without terminal η -expansions.

Proof: Obvious.

THEOREM 3.2: The function f is not monotonic.

Proof: Obvious from lemma 3.2.

Notice that also in the case in which Y is of type Σ as X , we can have $\mathcal{I}_L(X) \not\subseteq \mathcal{I}_L(Y)$. For example if

$$X = \lambda x_0 x_1 x_2. x_1(x_2 x_0) \quad \text{and} \quad Y = \lambda x_0 x_1 x_2. x_1(\lambda x_3. x_2 x_0 x_3)$$

we have that $X_L = \lambda z. z \mathbf{II}$ is a left inverse for X but not for Y .

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