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EXTENDED PRIMITIVE RECURSIVE FUNCTIONS (*)

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Abstract. — *In this paper we extend the notion of primitive recursive function to the case in which the domain is N^* and the range is a subset of N^*. We give a rigorous characterization of this class of functions and show differences with the classical primitive recursive functions.*

Résumé. — *Dans cette note nous étendons la notion de fonction primitive recursive à tous les cas dans lesquels le domaine est N^* et le codomaine est contenu en N^*. Nous donnons une caractérisation rigoureuse de cette classe de fonctions et montrons aussi des différences entre cette classe et celle des fonctions primitives récursives classiques.*

1. INTRODUCTION

Primitive recursive functions were first introduced as functions from N^r to N and then extended to the case of functions from N^r to N^s [1, 2, 3]. In this paper we propose a wider extension that has not yet been studied. We introduce the notion of sequence primitive recursive function as a function defined on sequences of naturals of variable length and having as value still sequences of the same type. This definition, a meaningful enlargement of the classical presentations, seems more adequate from a computer science point of view because, in many cases, the real programs work on non fixed length sequences as input and also give non fixed length sequences as output and therefore it is not possible to represent them with the more known formalizations of the recursive functions. On the other hand it is not a restriction to consider primitive recursive functions because it is well known that most functions computed in practice are indeed primitive recursive.

The aim of this paper is twofold. On one hand to find a class of functions from N^* to N^* sufficiently powerful to capture the notion of sequence primitive recursive function and on the other hand to study the main properties of this

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class, particularly paying attention to the differences with the classical presentations of primitive recursiveness.

The approach here presented was already proposed in [4] for the class of partial recursive functions and, for the reasons explained before, we intended, with this work, to give a further contribution to this new way of looking at recursion theory.

2. SEQUENCE PRIMITIVE RECURSIVE FUNCTIONS

NOTATIONS: In what follows we shall always be concerned with functions having domain $N^* = \bigcup_{i=0}^{\infty} N^i$ and range in N^*; we denote these functions with capital latin letters such as F, G, ... We call *sequence*, and denote with capital latin letters such as X, Y, Z a whatever element of N^*, included the empty sequence $\Lambda = N^0$; we denote with small latin letters as x, y, z a whatever element of N, the set of natural numbers, and small latin letters such as f, g, h the primitive recursive functions.

Being $X = x_1, \ldots, x_n$, $|X|$ will denote the length of the sequence, with the convention that $|\Lambda| = 0$.

As usual, we call N^+ the set $N - \{\Lambda\}$ and moreover we call L the null function:

$$L: \quad N^* \to \Lambda.$$

In the following we always call PR_n the class of primitive recursive functions of n variables (PR_1 the primitive recursive functions from N to N), and we consider as known all their classical properties, which we shall use whenever we need. In particular, we use the property that the characteristic functions of all relations which we shall introduce (such as equality) are primitive recursive. We also use the fact that, if $f \in PR_n$, then the following function:

$$g(y, x_1, \ldots, x_n) = f^y(x_1, \ldots, x_n) \doteq \underbrace{f(f(\ldots f}_{y \text{ times}}(x_1, \ldots, x_n))\ldots)$$

belongs to PR_{n+1}.

Gödel numbering

A well known property of primitive recursive functions which we shall use is the fact that it is possible to enumerate the set N^* in an one-to-one way, making use of primitive recursive functions and operators. We could make use of any primitive recursive bijective Gödel numbering; in particular we have chosen the surjective map of [5], with slight modifications in notation. Let $X \in N^*$,

$X = x_1, \ldots, x_n$; we call *Gödel number of* X, and write $x = Gn(X)$, the number defined in the following way:

$$x = \sum_{i=1}^{n} 2^{i-1+\sum_{j=1}^{i} x_j} = 2^{x_1} + 2^{x_1+x_2+1} + \ldots + 2^{x_1+x_2+\ldots+x_n+n-1},$$

with the convention that $Gn(\Lambda) = 0$.

We also need to define other functions: $lh(x)$ establishes the length of the coded sequence; $e(i, x)$ gives the exponent of 2 at the place i; and $T(i, x)$ gives the i-th element of the coded sequence for $i = 1, \ldots, lh(x)$. We shall make use of the following properties of the above functions:

$$T(i, Gn(x_1, \ldots, x_n)) = x_1, \qquad i = 1, \ldots, n,$$

$$Gn(T(1, x), \ldots, T(lh(x), x)) = x.$$

Finally, let us define the extension of the inverse function of Gn, which for simplicity we shall continue to call Gn^{-1} : $\forall x \in N$, if $Gn(x_1, \ldots, x_n) = x$ then:

$$x \xrightarrow{Gn^{-1}} x_1, \ldots, x_n$$

and:

$$\forall X \in N^* - N, \quad X \xrightarrow{Gn^{-1}} \Lambda.$$

DÉFINITION 2.1: A sequence function $F : N^* \to N^*$ is said to be a *sequence primitive recursive function* if and only if there exists a primitive recursive function $\varphi_F \in PR_1$ s. t. for every $X \in N^*$ if $X \xrightarrow{F} Y$, being $x = Gn(X)$ and $y = Gn(Y)$, then $\varphi_F(x) = y$.

We call *SPR* the class of all sequence primitive recursive functions.

Note that this definition naturally extends the notion of primitive recursive function of n variables.

PROPOSITION 2.2 (a): *Let* $f \in PR_n$. *Then* f *may be extended to a sequence primitive recursive function* F, *which we call* extension of f *in the following way:*

$$F = \begin{cases} f & in\ N^n, \\ L & in\ N^* - N^n, \end{cases}$$

where L *is the already defined null function.*

(*b*) *Let* $F \in SPR$ *s. t. for all* $X \in N^n$, $F(X) \in N$. *Then the* restriction f *of* F *to* N^n *is an element of* PR_n.

Proof: In fact:

(*a*)
$$\varphi_F(x) = \begin{cases} 2^{f(T(1,x),\ldots,T(lh(x),x))} & \text{if } lh(x)=n, \\ 0 & \text{if } lh(x) \neq n \end{cases}$$

and therefore $\varphi_F \in PR_1$ and $F \in SPR$.

(*b*) For all $X \in N^n$:

$$F(X) = F(x_1, \ldots, x_n) = T\left(1, \varphi_F\left(\sum_{i=1}^{n} 2^{i-1+\sum_{j=1}^{i} x_j}\right)\right)$$

and therefore the restriction $f \in PR_n$. □

3. THE CLASS $\mathscr{U}$

In the following, unless otherwise specified, X will be whatever element of N^*. Let us consider the following sequence functions:

$$O_r: \quad X \xrightarrow{O_r} X, 0,$$

$$S_r: \quad X, y \xrightarrow{S_r} X, y+1,$$

$$\Lambda \xrightarrow{S_r} \Lambda,$$

$$\leftarrow: \quad X, y \xrightarrow{\leftarrow} y, X,$$

$$\Lambda \xrightarrow{\leftarrow} \Lambda.$$

We now call the set $\mathscr{I} = \{O_r, S_r, \leftarrow\}$ the present set of *initial functions*.

Let us consider the following operators:

Composition: Given two sequence functions F and G, we call *composition* of F and G, and write $H = FG$, the sequence function H such that if:

$$X \xrightarrow{F} Y \xrightarrow{G} Z \qquad \text{then} \quad X \xrightarrow{H} Z$$

for all $X, Y, Z \in N^*$.

Repetition: Given a sequence function F, we call *repetition* of F, and write $H=[F]$, the sequence function H such that if $X \overset{F}{\rightarrow} X_1 \overset{F}{\rightarrow} \dots \overset{F}{\rightarrow} X_y$ then:

$$y, X \overset{H}{\rightarrow} X_y \qquad \text{if} \quad y \in N^+,$$

$$O, X \overset{H}{\rightarrow} X \qquad \text{and} \qquad \Lambda \overset{H}{\rightarrow} \Lambda.$$

DÉFINITION 3.1: We call $\mathscr{U}$ the smallest class of functions built up from $\mathscr{I}$ with a finite number of compositions and repetitions.

Examples 3.2: The following functions belong to the class $\mathscr{U}$:

3.2.1. $\begin{cases} O_l: \quad X \rightarrow O, X \\ \quad O_l = O_r \leftarrow \end{cases}$

3.2.2. $\begin{cases} \rightarrow: \quad y, X \rightarrow X, y \\ \qquad \Lambda \rightarrow \Lambda \\ \quad \rightarrow = O_r [S_r] \end{cases}$

3.2.3. $\begin{cases} S_l: \quad y, X \rightarrow y+1, X \\ \qquad \Lambda \rightarrow \Lambda \\ \quad S_l = \rightarrow S_r \leftarrow \end{cases}$

3.2.4. $\begin{cases} I: \quad X \rightarrow X \\ \quad I = \rightarrow \leftarrow \end{cases}$

3.2.5. $\begin{cases} \square_l: \quad y, X \rightarrow X \\ \qquad \Lambda \rightarrow \Lambda \\ \quad \square_l = [I] \end{cases}$

3.2.6. $\begin{cases} \square_r: \quad X, y \rightarrow X \\ \qquad \Lambda \rightarrow \Lambda \\ \quad \square_r = \leftarrow \square_l \end{cases}$

3.2.7. $\begin{cases} \leftrightarrow: \quad y, X, z \rightarrow z, X, y \\ \qquad t \rightarrow t \qquad \text{for} \quad t \in N \cup \{\Lambda\} \\ \leftrightarrow = \rightarrow O_l \leftarrow \leftarrow [\rightarrow S_l \leftarrow] \rightarrow \end{cases}$

3.2.8. $\begin{cases} D_l: \quad y, X \rightarrow y, y, X \\ \qquad \Lambda \rightarrow O \\ D_l = O_r O_r [S_r \leftarrow S_r \rightarrow] \leftarrow \leftarrow \end{cases}$

3.2.9.
$$\begin{cases} D_r: \quad X, y \to X, y, y \\ \Lambda \to O \\ D_r = \leftarrow D_l \to \to \end{cases}$$

3.2.10. Let k be a fixed number:

$$k_l: \quad X \to k, X$$
$$k_r: \quad X \to X, k$$

$$k_l = O_l \underbrace{S_l \ldots S_l}_{k \text{ times}}, \qquad k_r = k_l \to$$

3.2.11. $$\leftarrow^*: \quad k, X, Y \to Y, X$$

where the sequence Y has exactly k elements:

$$\leftarrow^* = [\leftarrow].$$

3.2.12. $$\Box_l^*: \quad k, Y, X \to X$$

where the sequence Y has exactly k elements:

$$\Box_l^* = [\Box_l]$$

3.2.13. $$U^*: \quad k, x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n$$
$$\to x_k, x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n$$

for $k = 1, \ldots, n$,

$$U^* = D_l [\to \leftrightarrow] D_r [\leftarrow \leftrightarrow] \leftarrow.$$

Note that we only described the behaviour of the functions 3.2.11, 3.2.12, 3.2.13 in the set that we were interested in.

We are now going to show that for every primitive recursive function f there exists a sequence function J_f which joins the result of the computation of the primitive recursive function to the left of the sequence, if the sequence has a length greater than or equal to the number of variables of the primitive recursive function; we do this without counting such a number.

THEOREM 3.3: *For all $n \in N^+$ and for every $f \in PR_n$ there exists a sequence function $J_f \in \mathscr{U}$ s. t. if $X \in N^n$ and $Y \in N^*$, then:*

$$X, Y \xrightarrow{J_f} f(X), X, Y.$$

Proof: By induction on the definition of primitive recursive function (we consider the classical one):

$$J_{Z^{(n)}} \in \mathcal{U} \qquad \text{where} \quad Z^{(n)}(x_1, \ldots, x_n)=0,$$

for all $n \in N^+$.

In fact $J_{Z^{(n)}} = O_l$.

$$J_S \in \mathcal{U} \qquad \text{where} \quad S(x)=x+1,$$

In fact $J_S = D_l S_l$.

$$J_{U_i^{(n)}} \in \mathcal{U} \qquad \text{where} \quad U_i^{(n)}(x_1, \ldots, x_i, \ldots, x_n)=x_i$$

for all $n \in N^+$ and $i=1, \ldots, n$.

In fact $J_{U_i^{(n)}} = i_l\ U^*$, where i_l is the function defined in 3.2.10.

Let $f \in PR_m$, $g_1, \ldots, g_m \in PR_n$ and $h \in PR_n$ s. t.:

$$h(x_1, \ldots, x_n)=f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n)).$$

By inductive hypothesis there exist $m+1$ functions $J_f, J_{g_1}, \ldots, J_{g_m} \in \mathcal{U}$ for which the theorem holds.

Then $J_h \in \mathcal{U}$. In fact:

$$J_h = J_{g_1} \rightarrow J_{g_2} \rightarrow \ldots \rightarrow J_{g_m} \rightarrow m_l \leftarrow^* J_f \rightarrow m_l \square_l^* \leftarrow.$$

Let $g \in PR_n$, $h \in PR_{n+2}$ and $f \in PR_{n+1}$ s. t.:

$$f(y, x_1, \ldots, x_n)=\begin{cases} g(x_1, \ldots, x_n) & y=0, \\ h(f(y-1, x_1, \ldots, x_n), y-1, x_1, \ldots, x_n), & y>0. \end{cases}$$

By inductive hypothesis there exist two functions $J_g, J_h \in \mathcal{U}$ for which the theorem holds. Then $J_f \in \mathcal{U}$. In fact:

$$J_f = \rightarrow J_g \rightarrow O_l \leftarrow \leftarrow [J_h \rightarrow \square_l S_l \leftarrow]. \qquad \square$$

Theorem 3.3. can be further strengthened:

THEOREM 3.4: *Let $f \in PR_n$. Let $\overline{\overline{F}}$ be a sequence function s. t., for any $X \in N^n$, $\overline{\overline{F}} = f$. Then $\overline{\overline{F}} \in \mathcal{U}$.*

Proof: In fact:

$$\overline{\overline{F}} = J_f\, n_l [\square_r]. \qquad \square$$

Theorem 3.4. can be generalized to primitive recursive functions having, as value, sequences of fixed length.

DEFINITION 3.5: A function $g : N^r \to N^s$ is said to be *fixed length sequence primitive recursive* if there exist s functions $g_1, g_2, \ldots, g_s \in PR_r$ s. t., for any r-tuple $x_1, \ldots, x_r \in N^r$:

$$g(x_1, \ldots, x_r) = g_1(x_1, \ldots, x_r), \ldots, g_s(x_1, \ldots, x_r) = y_1, \ldots, y_s.$$

THEOREM 3.6: *Let $g : N^r \to N^s$ be a fixed length sequence primitive recursive function. Let $\bar{\bar{G}}$ be a sequence function s. t., for any $X \in N^r$, $\bar{\bar{G}} = g$. Then $\bar{\bar{G}} \in \mathscr{U}$.*

Proof: In fact:

$$\bar{\bar{G}} = J_{g_1} \to J_{g_2} \to \ldots J_{g_s} \to r_l \square_l^*. \qquad \square$$

The interesting and nice properties of the class $\mathscr{U}$ lead to wonder whether the class $\mathscr{U}$ is a characterization of the set of all sequence primitive recursive functions. Although the choice was made analogously to the case of the classical primitive recursive functions, we can show that there exist sequence primitive recursive functions which do not belong to the class $\mathscr{U}$.

4. THE CLASS $\mathscr{P}$

To obtain a meaningful characterization of the class SPR, we need a function counting the length of an arbitrary sequence X. Despite its apparent simplicity, we are going to show that this function cannot belong to $\mathscr{U}$.

First of all, we can state a lemma which essentially shows that the "value" of every function belonging to $\mathscr{U}$ is bounded by the value of a suitable increasing primitive recursive function.

LEMMA 4.1: *For any function $F \in \mathscr{U}$ there exists a strictly increasing function $f_F \in PR_1$ s. t. for all $X = x_1, \ldots, x_n \in N^*$*
if $F(X) = Y = y_1, \ldots, y_m$, then $\bar{y} \leq f_F(\bar{x})$ where:

$$\bar{x} = \begin{cases} \text{Max}(x_1, \ldots, x_n) & \text{if } X \neq \Lambda, \\ O & \text{otherwise}, \end{cases}$$

$$\bar{y} = \begin{cases} \text{Max}(y_1, \ldots, y_m) & \text{if } X \neq \Lambda, \\ O & \text{otherwise}. \end{cases}$$

Proof: By induction on the definition of $\mathscr{U}$. As regards the initial functions O_r, S_r, $\leftarrow$, the corresponding primitive recursive function can be very easily found. In fact we have:

$$f_{O_r}(x)=f_{S_r}(x)=f_{\leftarrow}(x)=x+1.$$

We can now consider the case of the composition. Let $H=FG$, with $F, G\in\mathscr{U}$. By inductive hypothesis there exist f_F and f_G which verify the thesis. Then: $f_H(x)=f_G(f_F(x))$.

In fact if $X \xrightarrow{F} Y \xrightarrow{G} Z$, $X \xrightarrow{H} Z$, we have $\bar{y}\leqq f_F(\bar{x})$ and $\bar{z}\leqq f_G(\bar{y})$. Since f_G is strictly increasing:

$$\bar{z}\leqq f_G(f_F(\bar{x})).$$

Finally, let $H=[F]$. By inductive hypothesis there exists f_F for which the lemma holds. Then $f_H(x)=f_F^x(x)$.

In fact if $X \xrightarrow{F} X_1 \xrightarrow{F} \ldots \xrightarrow{F} X_y$:

$$y, X \xrightarrow{H} X_y, \qquad X\in N^*, \quad y\in N^+,$$

we have:

$$\bar{x}_1\leqq f_F(\bar{x}), \ \bar{x}_2\leqq f_F(\bar{x}_1), \ \ldots, \ \bar{x}_y\leqq f_F(\bar{x}_{y-1}).$$

Still, because of the strict increase of f_F:

$$\begin{aligned}\bar{x}_y \leqq f_F(f_F(\ldots f_F(\bar{x})\ldots))=f_F^y(\bar{x}) &\leqq f_F^y(\operatorname{Max}(y, x_1, \ldots, x_n))\\ &\leqq f_F(\operatorname{Max}(y, x_1, \ldots, x_n))^{\operatorname{Max}(y_1, x_1, \ldots, x_n)}\end{aligned}$$

The cases in which $y=0$ or in which H is to be applied to the empty sequence can be easily verified. □

THEOREM 4.2: *Let K be the following function:*

$$x_1, \ldots, x_n \xrightarrow{K} x_1, \ldots, x_n, n,$$

$$\Lambda \xrightarrow{K} O.$$

Then $K\notin\mathscr{U}$.

Proof: If $K \in \mathscr{U}$, then there would exist, by lemma 4.1., a strictly increasing function $f_K \in PR_1$ s. t. for all $X \in N^*$:

$$X \overset{K}{\rightarrow} X, \qquad |X| \quad \Rightarrow \quad \operatorname{Max}(\bar{x}, |X|) \leqq f_K(\bar{x}).$$

Instead, given an arbitrary sequence X, we can define a new sequence X' s. t.:

$$\operatorname{Max}(\bar{x}', |X'|) > f_K(\bar{x}').$$

Let $X = x_1, \ldots, x_n$. Then let $X' = x_1, \ldots, x_n, \underbrace{0, \ldots, 0}_{f_K(\bar{x})+1 \text{ times}}$.

The following inequalities lead to the contradiction:

$$\operatorname{Max}(\bar{x}', |X'|) \geqq f_K(\bar{x}) + 1 > f_K(\bar{x}) = f_K(\bar{x}'). \qquad \square$$

Since we have shown that K does not belong to the class $\mathscr{U}$, we need to define a new class containing K and $\mathscr{I}$.

DEFINITION 4.3: Let $\mathscr{I}' = \{ O_r, S_r, \leftarrow, K \}$ (the new set of initial functions). We call $\mathscr{P}$ the smallest class built up from $\mathscr{I}'$ with a finite number of compositions and repetitions.

We can now give some important examples of functions which are in $\mathscr{P}$ and not in $\mathscr{U}$.

4.3.1. $Gn \in \mathscr{P}$. In fact:

$$Gn = K O_l O_l O_l \leftarrow [D_l S_l S_l S_l S_l U^* \rightarrow \leftrightarrow J_\Sigma \rightarrow \square_l \square_l \leftarrow \leftarrow$$

$$J_\Sigma J_{\mathrm{Exp}_2} 5_l U^* J_\Sigma \rightarrow \square_l \square_l \square_l S_l \rightarrow \rightarrow \square_l \leftarrow \leftarrow \leftrightarrow \rightarrow \leftrightarrow \leftarrow \leftarrow] \square_l \square_l J_{lh} [\square_r],$$

where $lh(x)$ is the function already defined in paragraph 2, and $\Sigma : N \times N \rightarrow N$ and $\mathrm{Exp}_2 : N \rightarrow N$ are defined in the following way:

$$\Sigma : \; x, y \rightarrow x + y \qquad \text{and} \qquad \mathrm{Exp}_2 : \; x \rightarrow 2^x.$$

4.3.2. $Gn^{-1} \in \mathscr{P}$. In fact:

$$Gn^{-1} = K \leftarrow 1_l \, J_{\bar{\bar{D}}} [\square_l \, \square_l^* O_l] \, [\square_l J_{lh} \rightarrow 1_l \; \leftarrow [J_T \rightarrow S_l] \square_l \, \square_l],$$

where $\bar{\bar{D}} : N \times N \rightarrow N$ is so defined:

$$\bar{\bar{D}}(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

We are now able to show that the class $\mathscr{P}$ is a characterization of the class of sequence primitive recursive functions. In fact we have:

THEOREM 4.4: $SPR \equiv \mathscr{P}$.

Proof: (*a*) $SPR \subseteq \mathscr{P}$.

Let $F \in SPR$. Therefore by definition 2.1 there exists a function $\varphi_F \in PR_1$ s. t. if $X \xrightarrow{F} Y$ then $x \xrightarrow{F} y$, where $x = Gn(X)$ and $y = Gn(Y)$. Then $F = Gn\, J_{\varphi_F} \square_r\, Gn^{-1}$ and hence $F \in \mathscr{P}$.

(*b*) $\mathscr{P} \subseteq SPR$.

It is sufficient to prove that the initial functions belong to SPR and that applying the operations of composition and repetition to sequence primitive recursive functions we still obtain sequence primitive recursive functions.

(I) If $X \xrightarrow{O_r} Y$ where $Gn(X) = x$ and $Gn(Y) = y$ we have:

$$\varphi_{O_r}(x) = x + 2^{e(lh(x),\, x)+1} = y \qquad \text{and} \qquad \varphi_{O_r} \in PR_1.$$

Analogously:

$$\varphi_{S_r}(x) = x + 2^{e(lh(x),\, x)}, \qquad \varphi_{S_r} \in PR_1,$$

$$\varphi_{\leftarrow}(x) = \varphi_{\leftarrow}\left(\sum_{i=1}^{lh(x)} 2^{i-1+\sum_{j=1}^{i} T(j,\, x)}\right) = 2^{T(lh(x),x)} + \sum_{i=1}^{lh(x)-1} 2^{T(lh(x),x)+1+e(i,x)}, \qquad \varphi_{\leftarrow} \in PR_1.$$

(II) For the composition we have:

$$F,\, G \in SPR, \qquad H = FG \quad \Rightarrow \quad H \in SPR.$$

In fact:

$$\varphi_H(x) = \varphi_G(\varphi_F(x)).$$

(III) Finally, for the repetition we have:

$$F \in SPR, \qquad H = [F] \quad \Rightarrow \quad H \in SPR.$$

In fact:

$$\varphi_H(x) = \varphi_F^{T(1,x)}\left(\sum_{i=1}^{lh(x)-1} 2^{i-1+\sum_{j=1} T(j+1,x)}\right). \qquad \square$$

Since we have shown that it is possible to characterize the notion of sequence primitive recursive function in a simple and rigorous way, two research areas are the natural continuation of this paper. It would be interesting, on one hand, to look for characterizations of sequence elementary or lower elementary functions; and on the other hand, to introduce hierarchies of sequence primitive recursive functions.

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