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CODES, LANGUAGES AND MOL SCHEMES (*) (1)

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Abstract. — *The aim of this paper is to introduce and study a new class of DOL schemes, called MOL schemes. These are characterized by means of the OL languages they generate and by their preservation properties. Several special cases are investigated.*

1. INTRODUCTION.

Let X be an *alphabet* (a non-empty finite set) and let X^* be the *free monoid* generated by X . Let $X^+ = X^* - \{1\}$, where 1 is the empty word and let $lg(w)$ denote the *length* of the word $w \in X^*$. Any subset of X^* is called a *language*.

For any languages $A, B \subseteq X^*$, let $AB = \{xy \mid x \in A, y \in B\}$, $A^* = \bigcup_{i=0}^{\infty} A^i$ and $A^+ = \bigcup_{i=1}^{\infty} A^i$ (1-free iteration).

A OL scheme (see [1]) is an ordered pair (X, P) , where X is an alphabet and P (the set of productions) is a finite non-empty subset of $X \times X^*$ such that for any $a \in X$, there exists at least one $x \in X^*$ such that $(a, x) \in P$. Sometimes the notation $a \rightarrow x \in P$ will be used instead of $(a, x) \in P$. A OL scheme is *deterministic* if for every $a \in X$, the element $x \in X^*$ such that $a \rightarrow x \in P$ is unique and it is *propagating* if for every $a \rightarrow x \in P$, $x \neq 1$. The words DOL and PDOL will be used to represent the deterministic OL schemes and the propagating deterministic OL schemes respectively. If (X, P) is a OL scheme and if $x = a_1 a_2 \dots a_m$, $m \geq 0$, $a_i \in X$, $i = 1, 2, \dots, m$ and $y \in X^*$, then x is said to *directly generate* or *derive* y in (X, P) , denoted by $x \Rightarrow y$, if and only if there exist y_1, y_2, \dots, y_m such that $\{a_i \rightarrow y_i \mid i = 1, 2, \dots, m\}$ and $y = y_1 y_2 \dots y_m$. By this definition 1 directly derives y if and only if $y = 1$. The transitive and reflexive closure of the relation \Rightarrow is denoted by \Rightarrow^* . When $x \Rightarrow^* y$ then x is said to generate y in (X, P) . A OL *system* is a triple (X, P, w) , where (X, P) is a OL scheme and $w \in X^*$, called the *axiom* of (X, P, w) ; (X, P) is called the *scheme* of (X, P, w) . The language $L(X, P, w) = \{y \in X^* \mid w \Rightarrow^* y\}$ is called the OL *language* generated by

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(X, P, w) ; the notation $L(w)$ will also be used when there is no ambiguity concerning the scheme (X, P) . A language A is said to be a *OL language* if there exists a *OL system* (X, P, w) such that A is generated by (X, P, w) .

A mapping h of X^* into X^* such that $h(xy) = h(x)h(y)$ for all $x, y \in X^*$ is said to be a *homomorphism* of X^* into X^* or an *endomorphism* of X^* . If furthermore h is injective, i. e., if $h(x) = h(y)$ implies $x = y$, then h is said to be a *monomorphism*. If (X, P) is a *DOL scheme*, then the mapping h defined by $h(a_i) = x_i$, where $a_i \rightarrow x_i \in P$ determines a homomorphism of X^* into X^* . Conversely, every homomorphism h of X^* into X^* defines a *DOL scheme* (X, P) where $a_i \rightarrow x_i \in P$ if and only if $h(a_i) = x_i$. It follows that a *DOL scheme can be defined either by (X, P) or (X, h)* . In this paper we will use mainly the second definition. If (X, P, w) is a *DOL system*, then with the notation (X, h, w) , the *DOL language* $L(w)$ generated by the system is given by $L(w) = \{h^n(w) \mid n \geq 0\}$. If \mathcal{F} is a family of languages over X and if $h(A) \in \mathcal{F}$ for every $A \in \mathcal{F}$, then we say that the *DOL scheme* (X, h) *preserves* \mathcal{F} or that (X, h) is \mathcal{F} -*preserving*.

A *MOL scheme* (X, h) is a *DOL scheme* such that h is a *monomorphism*. It is immediate that a *MOL scheme* is always a *PDOL scheme*, but the converse is not true. Let us remark that a *DOL scheme* (X, h) such that $|X| = 1$ is always a *MOL scheme*, unless $h(X) = \{1\}$. A *DOL system* (X, h, w) such that (X, h) is a *MOL scheme* is called a *MOL system* and the language $L(X, h, w)$ is called a *MOL language*. The purpose of this paper is to establish some properties of the *MOL schemes*. In section 2, we characterize *MOL schemes* by using the properties of the *OL languages* generated by their associated *OL systems* and we give a biological interpretation of some of these results. In section 3, the characterization of *MOL schemes* is done by considering some classes of languages which they preserve, and the last section is concerned mainly with the study of particular classes of *MOL schemes*.

2. MOL SCHEMES AND LANGUAGES

PROPOSITION 1: *Let (X, h) be a MOL scheme. If $L(X, h, w_1) \cap L(X, h, w_2) \neq \emptyset$, then either $L(X, h, w_1) \subseteq L(X, h, w_2)$ or vice versa.*

Proof: There exist $m, n \geq 0$ such that $h^m(w_1) = h^n(w_2)$. If $m = n$, then $w_1 = w_2$ and $L(w_1) = L(w_2)$. Let $m < n$, $n = m + k$, $k \geq 1$. Then $h^m(w_1) = h^{m+k}(w_2)$ and $h^m(w_1) = h^m(h^k(w_2))$. Hence $w_1 = h^k(w_2)$ and $L(w_1) \subseteq L(w_2)$. #

Let us remark that if (X, h) is a *MOL scheme*, then $L(X, h, w_1) \subseteq L(X, h, w_2)$ if and only if $w_1 = h^k(w_2)$ for some $k \geq 0$.

PROPOSITION 2: *A PDOL scheme (X, h) is a MOL scheme if and only if $L(X, h, w_1) \cap L(X, h, w_2) \neq \emptyset$, $w_1, w_2 \in X^+$, implies either $L(X, h, w_1) \subseteq L(X, h, w_2)$ or vice versa.*

Proof: Necessity. This is Proposition 1. *Sufficiency.* Suppose h is not injective. Then there exist $v, w \in X^+$, $v \neq w$, such that $h(v) = h(w)$. It follows then that $L(v) \cap L(w) \neq \emptyset$ and hence either $L(v) \subseteq L(w)$ or $L(w) \subseteq L(v)$. Let us suppose $L(w) \subseteq L(v)$. Then $h^k(v) = w$ for some $k \geq 1$ and $h^{k+1}(v) = h(w) = h(v)$. Let $X(w) = \{x \mid x \in X, x \text{ is a subword of } w\}$. Since $h^{k+1}(v) = h(v)$, then $h^k(w) = w$ and $lg(h(x)) = 1$ for every $x \in X(w) \subseteq X$.

We claim that if $x, y \in X(w)$ and $h(x) = h(y)$, then $x = y$. Suppose on the contrary $x \neq y$ and $h(x) = h(y) = a \in X$. Then $h(xyx) = a^3 = h(yxy)$ and $L(xyx) \cap L(yxy) \neq \emptyset$. Hence $L(xyx) \subseteq L(yxy)$ or *vice versa*. Suppose the first case: then since $xyx \neq yxy$, we have $xyx = h^k(yxy)$ for some $k \geq 1$. Therefore, $h^k(yxy) = u^3$ for some u and $xyx = u^3$, a contradiction. The second case is also impossible.

Now if $X(v) \subseteq X(w)$, then $h(v) = h(w)$ implies $v = w$, a contradiction. Hence $X(v) \not\subseteq X(w)$ and there exists $z \in X$ such that $z \in X(v)$, $z \notin X(w)$. Therefore $v = v_1 z v_2$ and $h(v) = h(v_1) h(z) h(v_2)$. Since $h(x) \in X$ for $x \in X(w)$ and since $h(v) = h(w)$, it follows then that w can be written in the form $w = y_1 y y_2$ where $h(y) = h(z) = d$. We have $h(zyz) = d^3 = h(yzy)$ and $L(zyz) \cap L(yzy) \neq \emptyset$. Hence $L(zyz) \subseteq L(yzy)$ or *vice versa*. Suppose the first case: then $zyz = h^k(yzy)$, for some $k \geq 1$ and $zyz = t^3$ for some $t \in X^+$. Since $z \notin X(w)$, then $z \notin X(y)$ and the equality $zyz = t^3$ is impossible. By the same argument we can show that the second case is also impossible. #

The following biological interpretation can be given of the preceding proposition. Let us suppose that we have two organisms which are developing according to the same DOL scheme (X, h) . Then the scheme (X, h) is a MOL scheme if and only if either of these two organisms have a completely different development or one of them can be considered as the descendant of the other.

Let (X, h) be a DOL scheme. Define on X^* the relation H by $x H y \Leftrightarrow h^m(x) = h^n(y)$ for some $m, n \geq 0$. This relation is clearly an *equivalence* relation. Let us denote by $H(x)$ the class of x . Every OL language with scheme (X, h) is contained in a class of H .

If (X, h) is a MOL scheme, then $h^m(w_1) = h^n(w_2)$, $m \leq n$, implies $w_1 = h^{n-m}(w_2)$. Therefore $v H w$ if and only if there exists $n \geq 0$ such that either $v = h^n(w)$ or $w = h^n(v)$.

PROPOSITION 3: *A PDOL scheme (X, h) is a MOL scheme if and only if every class of H is a OL language.*

Proof: Necessity. Let A be a class of H . If $1 \in A$, then $A = \{1\}$ and $A = L(X, h, 1)$. Let $1 \notin A$ and let B be the set of the words of minimal length in A . For every pair $w_1, w_2 \in B$, then either $w_1 = h^n(w_2)$ or $w_2 = h^n(w_1)$ for some $n \geq 0$. Since B is finite, there exists $v \in B$ such that, for any $w \in B$, $w = h^n(v)$ for some $n \geq 0$. Let $u \in A$, $u \notin B$; then $h^m(u) = h^n(v)$ for some $m, n \geq 0$. Since (X, h) is propagating, then $m \leq n$ and $u = h^{n-m}(v)$. Therefore $A = L(X, h, v)$.

Sufficiency. Suppose $L(X, h, w_1) \cap L(X, h, w_2) \neq \emptyset$ with $w_1, w_2 \in X^+$. Then, since each OL language with scheme (X, h) is contained in a class of H , $L(X, h, w_1)$ and $L(X, h, w_2)$ are contained in the same class A of H . But $A = L(X, h, v)$ for some $v \in X^+$. Hence $w_1 = h^m(v)$, $w_2 = h^n(v)$ for some $m, n \geq 0$. Suppose $n = m+k$, $k \geq 0$. Then $w_2 = h^{m+k}(v) = h^k(w_1)$. Therefore $L(X, h, w_2) \subseteq L(X, h, w_1)$ and (X, h) is a MOL scheme by Proposition 2. #

A OL language L with DOL scheme (X, h) is said to be *maximal* if the inclusion $L \subseteq L'$, where L' is a OL language with the same scheme (X, h) , implies $L = L'$.

If (X, h) is a PDOL scheme, it is easy to see that every OL language with scheme (X, h) is contained in at least a maximal one. The following example shows that in general there can be several distinct maximal OL languages containing the same OL language.

Let $X = \{a, b\}$, $h(a) = ab$, $h(b) = ab$. Then $L(X, h, a)$ and $L(X, h, b)$ are distinct maximal OL languages containing the OL language $L(X, h, ab)$ with the PDOL scheme (X, h) .

PROPOSITION 4: *A PDOL scheme (X, h) is a MOL scheme if and only if every OL language L with scheme (X, h) is contained in a unique maximal OL language with the same scheme.*

Proof: Necessity. Since (X, h) is a PDOL scheme, L is contained in at least one maximal OL language. Let M_1 and M_2 be two maximal OL languages containing L . Then $L \subseteq M_1 \cap M_2$, and by Proposition 2, $M_1 \subseteq M_2$ or $M_2 \subseteq M_1$. Hence $M_1 = M_2$.

Sufficiency. Let A be a class of H , $A \neq \{1\}$ and let $v \in A$. Then $L(X, h, v) \subseteq A$ and there is a unique maximal OL language M such that $L(X, h, v) \subseteq M$. It is immediate that $M \subseteq A$. Suppose $M \neq A$. Then there exists $w \in A$, $w \notin M$. Since $v H w$, then $h^m(v) = h^n(w) = u$ for some $n, m \geq 0$. Therefore $u \in L(X, h, v) \subseteq M$ and $u \in L(X, h, w) \not\subseteq M$. Let M' be the unique maximal OL language containing $L(X, h, w)$. Since $u \in L(X, h, w)$, we have $L(X, h, u) \subseteq M$ and $L(X, h, u) \subseteq M'$, a contradiction. Hence $M = A$ and every class of H is a OL language. By Proposition 3, it follows then that (X, h) is a MOL scheme. #

3. CODES AND MOL SCHEMES

A non-empty language $A \subseteq X^+$ is said to be a *code* if $a_1 a_2 \dots a_n = b_1 b_2 \dots b_m$, $m \geq 1, n \geq 1$ and $a_i, b_j \in A$ implies $n = m$ and $a_i = b_i, i = 1, 2, \dots, n$. A code A is called a *prefix code* if $A \cap AX^+ = \emptyset$. (see [4]). The relation p_c defined on X^* by $x p_c y$ if and only if $y = xu = ux$ for some $u \in X^*$ is a partial order and a

non-empty language $A \subseteq X^+$ is called ρ_c -independent if for any $x, y \in A$, $x \rho_c y$ implies $x = y$ (see [8]).

PROPOSITION 5: Every DOL scheme (X, h) that is code preserving is propagating.

Proof: For any $a \in X$, $h(a) \neq 1$, because $\{a\}$ is a code but $\{1\}$ is not. #

PROPOSITION 6: A DOL scheme (X, h) is a code preserving scheme if and only if (X, h) is a MOL scheme.

Proof: Suppose first that (X, h) is code preserving. Then $h(X)$ is a code. Moreover, if $a_i, a_j \in X, a_i \neq a_j$, then $h(a_i) \neq h(a_j)$. Indeed, if $h(a_i) = h(a_j) = c$, then $A = \{a_i, a_j\}$ is a code but not $h(A) = \{c, c\}$, a contradiction. Now if h is not injective, then there exist $x \neq y, x, y \in X^+$ such that $h(x) = h(y)$. Let

$$x = x_1 \dots x_m, \quad y = y_1 y_2 \dots y_n, \quad m \geq 1, \quad n \geq 1 \quad \text{and} \quad x_i, y_j \in X;$$

then

$$h(x_1) \dots h(x_m) = h(x) = h(y) = h(y_1) \dots h(y_n).$$

Since $h(X)$ is a code, we have $m = n$ and $h(x_i) = h(y_i), i = 1, 2, \dots, n$. This implies that $x_i = y_i, i = 1, 2, \dots, n$ and $x = y$ holds, a contradiction.

Suppose now that (X, h) is a MOL scheme and that (X, h) is not code preserving. Then there exists a code A over X such that $h(A)$ is not a code, and therefore $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m \in A, x_1 \neq y_1$ such that

$$h(x_1) \dots h(x_n) = h(y_1) \dots h(y_m).$$

This implies that

$$h(x_1 \dots x_n) = h(y_1 \dots y_m).$$

Since h is injective, we have $x_1 \dots x_n = y_1 \dots y_m$. It follows then that $x_1 = y_1$ and since A is, by assumption, a code, a contradiction. #

PROPOSITION 7: A DOL scheme (X, h) is a MOL scheme if and only if $h(X)$ is a code and $|h(X)| = |X|$.

Proof: Necessity. This follows immediately from Proposition 6.

Sufficiency. Suppose that h is not injective. Since $h(X)$ is a code, then $1 \notin h(X)$ and there exist $x, y \in X^+, x \neq y$, such that $h(x) = h(y)$. Let

$$x = x_1 x_2 \dots x_m, \quad y = y_1 y_2 \dots y_n, \quad x_i, y_j \in X, \quad m \geq 1, \quad n \geq 1.$$

Then

$$h(x_1 x_2 \dots x_m) = h(y_1 y_2 \dots y_n)$$

and

$$h(x_1) h(x_2) \dots h(x_m) = h(y_1) h(y_2) \dots h(y_n).$$

Since $h(X)$ is a code by assumption, we have $n = m$ and

$$h(x_i) = h(y_i) \quad \text{for all } i = 1, 2, \dots, n.$$

Since $|h(X)| = |X|$, and X is finite we have $x_i = y_i$ for all $i = 1, 2, \dots, n$. Thus $x = y$, a contradiction. Hence h is injective and (X, h) is a MOL scheme. $\#$

If $A \subseteq X^+$, $A \neq \emptyset$, then A is ρ_c -independent if and only if every pair of two distinct elements from A form a code (see [8]). We note that for any $x, y \in X^+$, $\{x, y\}$ is a code if and only if $xy \neq yx$.

PROPOSITION 8: *A DOL scheme (X, h) is a MOL scheme if and only if (X, h) preserve the ρ_c -independent languages.*

Proof: Necessity. Let $A \subseteq X^+$ be a ρ_c -independent language. Suppose $h(A)$ is not ρ_c -independent. Then there exist $x, y \in A$, $x \neq y$ such that $\{h(x), h(y)\}$ is not a code.

This implies that $h(x)h(y) = h(y)h(x)$ and $h(xy) = h(yx)$ holds. Since h is injective by assumption, we have $xy = yx$. This contradicts the fact that A is a ρ_c -independent language.

Sufficiency. Suppose that h is not injective. Then there exist $x, y \in X^+$, $x \neq y$, such that

$$h(x) = h(y) = z, \quad z \neq 1, \quad \text{and} \quad h(xy) = h(yx) = z^2.$$

Now if $xy = yx$, then

$$x = p^n, \quad y = p^m \quad \text{for some } p \in X^+, \quad \text{and} \quad m \geq 1, \quad n \geq 1.$$

Since $[h(p)]^n = h(x) = h(y) = [h(p)]^m \neq 1$, we have $n = m$, a contradiction. On the other hand, if $xy \neq yx$, then $\{x, y\}$ is a code. The set $A = \{x, xy\}$ is then a code, but $h(x) = z$, $h(xy) = z^2$ and so $\{h(x), h(xy)\}$ is not a code, again a contradiction. $\#$

4. SPECIAL CLASSES OF MOL SCHEMES

In this section, we consider MOL schemes which preserve special classes of languages.

Let us recall that a language A over X is said to be a *right power-bounded language* if there exists a positive integer n such that $yx^m \in A$, $x \neq 1$ implies that $m \leq n$ (see, [9]).

PROPOSITION 9: *Let (X, h) be a DOL scheme such that $h(X) \neq \{1\}$. If (X, h) is a scheme which preserves the regular right power-bounded languages, then (X, h) is a MOL scheme.*

Proof: First we show that for any $a \in X$, $h(a) \neq 1$. Suppose $h(a) = 1$; then there exists $b \in X$ such that $h(b) \neq 1$, since $h(X) \neq \{1\}$ by assumption. The

language $A = \{b^n a \mid n \geq 1\}$ is a regular right power-bounded language, but $h(A) = \{h(b)^n \mid n \geq 1\}$ is not a right power-bounded language, a contradiction. Thus $h(a) \neq 1$, for all $a \in X$.

Now suppose h is not injective. Then $h(x) = h(y)$, $x \neq y$, for some $x, y \in X^+$. We can choose x and y such that $x = az_1$, $y = bz_2$, $a \neq b$, $a, b \in X$ and $z_1, z_2 \in X^*$.

Then $h(x) = h(a)h(z_1) = h(y) = h(b)h(z_2)$. We may assume $lg(h(a)) \leq lg(h(b))$. Let $h(a) = v$, $h(b) = w$. Then $w = vu$ for some $u \in X^*$. The language $A = \{b^n a \mid n \geq 1\}$ is a regular right power-bounded language, but $h(A) = \{(vu)^n v \mid n \geq 1\} = \{v(uv)^n \mid n \geq 1\}$ is not a right power-bounded language, a contradiction. #

The converse of this Proposition is false. For example, let (X, h) be a DOL scheme such that $X = \{a, b\}$, $h(a) = ba$, $h(b) = b$. Let $A = \{a^n b \mid n \geq 1\}$. Then $|h(X)| = |X|$ and $h(X)$ is a code. Hence by Proposition 7, (X, h) is a MOL scheme. But $h(A) = \{(ba)^n b \mid n \geq 1\} = \{b(ab)^n \mid n \geq 1\}$ is not a right power-bounded language while A is.

PROPOSITION 10: Let (X, h) be a DOL scheme such that $|h(X)| = |X|$. Then $h(X)$ is a prefix code if and only if (X, h) is a scheme which preserves the prefix codes.

Proof: Sufficiency. Trivial.

Necessity. Let A be a prefix code. We have to show that $h(A)$ is also a prefix code. The case $|h(A)| = 1$ is trivial. Suppose that $|h(A)| \geq 2$ and let $p \neq q \in h(A)$. Then there exist $u, v \in A$ such that $h(u) = p$, $h(v) = q$. Let $u = u_1 u_2 \dots u_n$, $v = v_1 v_2 \dots v_m$, $u_i, v_j \in X$. Since $u_1 u_2 \dots u_n = u \neq v = v_1 v_2 \dots v_m$, there exists $k \geq 1$ such that $u_k \neq v_k$ and $u_i = v_i$ for all $i < k$. Since

$$p = h(u) = h(u_1) \dots h(u_{k-1})h(u_k)h(u_{k+1} \dots u_n),$$

$$q = h(v) = h(v_1) \dots h(v_{k-1})h(v_k)h(v_{k+1} \dots v_m), \quad h(u_i) = h(v_i)$$

for all $i < k$ and $\{h(u_k), h(v_k)\}$ a prefix code by assumption, then $\{p, q\}$ is a prefix code. Therefore $h(A)$ is a prefix code. #

A word $w \in X^+$ is called a primitive word if $w = p^n$, $p \in X^+$, implies $n = 1$. It is well known that for any $x \in X^+$, there exists a unique primitive word p and $n \geq 1$ such that $x = p^n$. Let $Q = \{p \in X^+ \mid p \text{ is a primitive word}\}$, $Q^{(1)} = Q \cup \{1\}$ and $Q^{(i)} = \{p^i \mid p \in Q\}$, $i \geq 2$. Then $X^* = \bigcup_{i=1}^{\infty} Q^{(i)}$ and $Q^{(i)} \cap Q^{(j)} = \emptyset$ if $i \neq j$ (see [3]). If $x = p^n$, $p \in Q$, then $\sqrt{x} = p$ is called the root of x . In particular $\sqrt{1} = 1$. A language $A \subseteq X^*$ is called pure if for any $x \in A^*$, $\sqrt{x} \in A^*$.

A language $A \subseteq X^*$ is called noncounting (left-noncounting) if there exists $k \geq 1$ such that $ux^k v \in A$ if and only if $ux^{k+1} v \in A$, ($x^k v \in A$ if and only if

$x^{k+1}v \in A$) for all $u, x, v \in X^*$. A language $A \subseteq X^*$ is said to be a *power-separating language* if there exists $k \geq 1$, called the order of A , such that for any $x \in X^*$ either $x^k x^* \subseteq A$ or $x^k x^* \cap A = \emptyset$. Every noncounting language is left-noncounting and every left-noncounting language is power-separating, but the converse is not true (see [6], [7]).

In [5], Restivo has shown that a finite code $A \subseteq X^*$ is pure if and only if A^* is a noncounting language. In order to extend this result, let us recall that a language $A \subseteq X^*$ is a code if and only if $f \in X^*$, $fA^* \cap A^* \cap A^*f \neq \emptyset$ implies $f \in A^*$. From this, it follows that if A is a code, then x^n and $x^{n+r} \in A^*$ imply $x^r \in A^*$.

PROPOSITION 11: *Let $A \subseteq X^*$ be a finite code. Then the following are equivalent:*

- (1) A is pure;
- (2) A^* is a power-separating language;
- (3) A^* is a left-noncounting language.

Proof: (1) implies (3). Suppose A is pure. Then A^* is a noncounting language (see [5]) and hence a left-noncounting language.

(3) implies (2). Immediate.

(2) implies (1). Suppose that A is not pure. Then there exists a word $x \in A^*$ such that $x = p^k$, $k > 1$ and $p \notin A^*$. Thus $p^n \in A^*$ for all $n = kr$, $r \geq 1$. Since A is a code by assumption and since $p \notin A^*$, then $p^{n+1} \notin A^*$. This implies that A^* is not a power-separating language. #

A DOL scheme (X, h) is said to be a scheme *preserving the primitive words*, if for any primitive word $p \in X^+$, $h(p)$ is a primitive word. i. e., if $h(Q) \subseteq Q$.

PROPOSITION 12: *Every MOL scheme (X, h) such that $h(X)$ is a pure code, preserves the primitive words.*

Proof: Let $g \in Q$. Then $h(g) = p^n \in [h(X)]^* \subseteq X^*$, where $p \in Q$. Since $h(X)$ is pure by assumption, we have $p \in [h(X)]^*$. It follows then that for some $x \in X^*$, $h(x) = p$ and $h(x^n) = p^n = h(g)$. Since (X, h) is a MOL scheme, then h is injective and $g = x^n$. Since $g \in Q$, we have $n = 1$. Thus $h(g)$ is a primitive word. #

The MOL scheme (X, h) , where $X = \{a, b\}$ and $h(a) = ab$, $h(b) = ba$, is an example of a MOL scheme preserving the primitive words.

PROPOSITION 13: *Every MOL scheme (X, h) such that $h(X)$ is a pure code, preserves the pure languages.*

Proof: Let A be a pure language and let $p^n \in [h(A)]^*$, $p \in Q$. Then there exists $x \in A^*$ such that $h(x) = p^n$ and $x = q^m$, $q \in Q$. This implies that $p^n = h(q^m) = [h(q)]^m$. Since $h(X)$ is a pure code, then by Proposition 12, $h(q)$ is a primitive word. Hence $n = m$ and $p = h(q)$. Since A is pure, then

$x = q^m \in A^*$ implies that $q \in A^*$ and $p = h(q) \in [h(A)]^*$. Therefore $h(A)$ is pure. #

PROPOSITION 14: *Every MOL scheme, such that $h(X)$ is a pure code, preserves the power-separating languages.*

Proof: Since $h(X)$ is a pure code, then by Proposition 11, $[h(X)]^* = h(X^*)$ is a power-separating language, say of order m . Then, by definition, for any $x \in X^*$, either $x^m x^* \subseteq h(X^*)$ or $x^m x^* \cap h(X^*) = \emptyset$. Now let A be any power-separating language of order n . We will show that $h(A)$ is a power-separating language of order nm . Let $x \in X^*$, $x \neq 1$. If $x^m x^* \cap h(X^*) = \emptyset$, then $x^{nm} x^* \cap h(A) = \emptyset$. Now suppose that $x^m x^* \subseteq h(X^*)$. Then there exists $y \in X^*$ such that $h(y) = x^m$. Let $y = p^r$, $x = q^s$, $r, s \geq 1$, $p, q \in Q$. Then $[h(p)]^r = h(y) = x^m = q^{sm}$. Since $h(X)$ is pure, then by Proposition 12, $h(p)$ is primitive and $h(p) = q$, $r = sm$.

If $p^n p^* \subseteq A$, then

$$p^{nms} p^* \subseteq A \quad \text{and} \quad h(p^{nms} p^*) = [h(p)]^{nms} [h(p)]^* = q^{nms} q^* = x^{nm} q^* \subseteq h(A).$$

This implies that $x^{nm} x^* \subseteq h(A)$, because $x^* \subseteq q^*$.

If $p^n p^* \cap A = \emptyset$, then $p^n p^* \subseteq \bar{A} = X^* - A$ and \bar{A} is also a power-separating language of order n . By using the same argument as above, it can be shown that $x^{nm} x^* \subseteq h(\bar{A})$. Since h is injective, then $h(A) \cap h(\bar{A}) = \emptyset$, and therefore $x^{nm} x^* \cap h(A) = \emptyset$.

It follows then that $h(A)$ is a power-separating language of order nm . #

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