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Remarks on DOL growth sequences


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REMARKS ON DOL GROWTH SEQUENCES (*)

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Abstract. — Two theorems are given characterizing the position of DOL and PDOL growth sequences among \( N \)-rational sequences.

1. INTRODUCTION

A DOL system or a deterministic context-independent Lindenmayer system consists of an initial word \( \omega \) and a set of productions \( x \rightarrow \delta(x) \) which give for any letter \( x \) and thus also for any word a unique successor. The growth sequence of a DOL system is the sequence formed by the lengths of the words \( \omega, \delta(\omega), \delta^2(\omega), \ldots \). DOL sequences have been investigated e. g. in Paz and Salomaa [6], Salomaa [8], Vitányi [10], Ruohonen [7] and Karhumäki [4].

A sequence \( (r_n) \) is called \( N \)-rational if it can be represented in the form \( r_n = PM^n Q \) where \( P \) is a row vector, \( M \) is a square matrix, \( Q \) is a column vector and the entries of \( P, M \) and \( Q \) are natural numbers. (The name \( N \)-rational comes from the general theory of rational series founded by M. P. Schützenberger.) Now it is easy to see that a DOL sequence is \( N \)-rational; in fact it has a representation \( PM^n Q \) where \( Q \) consists merely of ones.

If a DOL sequence is not terminating, i. e. if \( r_n \neq 0 \) for every \( n \), and if \( L \) is the largest of the lengths of the words \( \delta(x) \) then obviously \( r_{n+1}/r_n \leq L \) for every \( n \). If the system under consideration is such that \( \delta(x) \) is always a non-empty word then this system is called a PDOL system and its growth sequence is called a PDOL sequence. Obviously a PDOL sequence is non-decreasing.

The goal of this paper is to illustrate the position of DOL and PDOL sequences among \( N \)-rational sequences. It will be seen that the satisfaction of an inequality \( r_{n+1}/r_n \leq L \) is characteristic for DOL sequences. Further it will be seen that it is not the non-negativity but the \( N \)-rationality of the sequence \( (r_{n+1} - r_n) \) that makes a DOL sequence to be a PDOL sequence.

2. PRELIMINARIES

A DOL system is at triple \( G = (X, \delta, \omega) \) where \( X = \{x_1, \ldots, x_k\} \) is an alphabet, \( \delta \) is an endomorphism of the free monoid \( X^* \) and \( \omega \in X^* \). The mapping \( \delta \) is usually given by writing the productions \( x_i \rightarrow \delta(x_i) \) and the

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word $\omega$ is called the axiom. If $\delta(x_i) \neq \lambda$ for each $i$ then $G$ is called a PDOL system. The function

$$f_G(n) = \lg(\delta^n(\omega)), $$

where $\lg$ means word length is called the growth function of $G$.

A pair $(X, \delta)$ where $X$ and $\delta$ are as above is called a DOL scheme. Introducing the axiom vector

$$P = (\lg_1(\omega), \ldots, \lg_k(\omega))$$

and the growth matrix

$$M = \begin{pmatrix}
\lg_1(\delta(x_1)) & \ldots & \lg_1(\delta(x_k)) \\
\vdots & \ddots & \vdots \\
\lg_k(\delta(x_1)) & \ldots & \lg_k(\delta(x_k))
\end{pmatrix}$$

of $G$ (here $\lg_j$ denotes the number of letters $x_j$) we obtain

$$(\lg_1(\delta^n(\omega)), \ldots, \lg_k(\delta^n(\omega))) = PM^n$$

and

$$f_G(n) = PM^n(1, \ldots, 1)^T.$$ 

A sequence $(r(n))$ is called $\mathbb{Z}$-rational (resp. $\mathbb{N}$-rational) if

$$r(n) = PM^n Q = (p_1, \ldots, p_k)
\begin{pmatrix}
m_{11} & \ldots & m_{1k} \\
\vdots & \ddots & \vdots \\
m_{k1} & \ldots & m_{kk}
\end{pmatrix}
\begin{pmatrix}
q_1 \\
\vdots \\
q_k
\end{pmatrix},$$

where all the entries are integers (resp. non-negative integers). If now $G$ is a DOL system (resp. a PDOL system) then by the above $(f_G(n))$ is a special $\mathbb{N}$-rational sequence called a DOL sequence (resp. a PDOL sequence).

It is known (Schützenberger [9]) that a sequence $(r(n))$ is $\mathbb{Z}$-rational ($\mathbb{N}$-rational) iff the series $\sum r(n) x^n$ is $\mathbb{Z}$-rational ($\mathbb{N}$-rational). If now $\sum r(n) x^n$ is a non-polynomial $\mathbb{N}$-rational series then a theorem of Berstel [1] concerning its poles tells the following: there are a natural number $p$, algebraic numbers $A, A_1, \ldots, A_s (A > 0, | A_j | < A, s \geq 0)$ and polynomials $H_0, \ldots, H_{p-1}, h_1, \ldots, h_s$ such that

$$r(i + np) = H_i(i + np) A^{i+np} + \sum_{j=1}^s h_j(i + np) A_j^{i+np}$$

for large values of $n$ ($i = 0, \ldots, p - 1$). In the case of a DOL sequence the polynomials $H_i$ must have a common degree $l$ because the quotients $r(n+1)/r(n)$ are bounded from above. We shall say that $(r(n))$ has the growth order $n^l A^n$. 

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3. THE PDOL SEQUENCES

**Lemma 1:** If \( (f(n)) = (PM^n Q) \) is an \( N \)-rational sequence and \( P \) has positive entries then \( (f(n)) \) is a DOL sequence.

**Proof:** Let \( G = (\{ x_1, \ldots, x_k \}, \delta, \omega) \) be a DOL system with axiom vector \( Q^T \) and growth matrix \( M^T \). Define \( G' = (\{ x_1, \ldots, x_k, x \}, \delta', \omega') \) where

\[
\delta'(x_i) = \delta(x_i) x^{1_{\Omega_1}}(\delta(x_i))(p_1 - 1) + \ldots + 1_{\Omega_k}(\delta(x_i))(p_k - 1),
\]

\[
\omega' = \omega x^{1_{\Omega_1}}(\omega)(p_1 - 1) + \ldots + 1_{\Omega_k}(\omega)(p_k - 1).
\]

Then obviously \( f(n) = Q^T (M^T)^n P^T = f_G(n) \).

**Theorem 1:** Let \( (r(n)) \) be an \( N \)-rational sequence. Then we can find natural numbers \( m \) and \( p \) and DOL sequences \( (d_0(n)), \ldots, (d_{p-1}(n)) \) such that

\[
r(m + i + np) = d_i(n) \quad (i = 0, \ldots, p - 1).
\]

**Proof:** Let \( r(n) = PM^n Q \) and let \( G = (X, \delta, \omega) \) be a DOL system with axiom vector \( P \) and growth matrix \( M \). Denote by \( X_n \) the set of letters occurring in \( \delta^n(\omega) \). Then we can find numbers \( m \) and \( p \) such that \( X_{m+i} = X_{m+i+np} \). We may of course suppose that no \( y_n \) is empty.

Introduce now the DOL systems

\[
G_i = (X_{m+i}, \delta^p, \delta^{m+i}(\omega)) \quad (i = 0, \ldots, p - 1)
\]

whose axiom vectors and growth matrices are denoted by \( P_i \) and \( M_i \). Then obviously

\[
r(m + i + np) = P_i M^n_i Q_i,
\]

where \( Q_i \) is composed of those entries of \( Q \) corresponding to letters of \( X_{m+i} \).

By lemma 1 we may define \( d_i(n) = P_i M^n_i Q_i \).

**Theorem 2:** The following conditions are equivalent for a sequence \( (r(n)) \):

(i) \( (r(n)) \) is a PDOL sequence not identically zero;

(ii) \( r(0) \) is a positive integer and the sequence \( (s(n)) = (r(n+1) - r(n)) \) is \( N \)-rational.

**Proof:** Suppose (i) holds. If now \( (r(n)) \) corresponds to a PDOL system \( G = (\{ x_1, \ldots, x_k \}, \delta, \omega) \) then

\[
s(n) = \sum_{i=1}^{k} \lg_i (\delta^n(\omega))(\lg(\delta(x_i)) - 1)
\]

and each of the sequences \( (\lg_i (\delta^n(\omega))) (i = 1, \ldots, k) \) is \( N \)-rational.
Suppose then that (ii) holds. Write according to theorem 1
\[ s(m+i+np) = d_i(n) \quad (i = 0, \ldots, p - 1) \]
where \((d_i(n))\) corresponds to a system \(G_i = (X_i, \delta_i, \omega_i)\). Assuming that the alphabets \(X_i\) are mutually disjoint we construct the PDOL system \(G = (X, \delta, \omega)\) where
\[
X = \left( \bigcup_{i = 0}^{p-1} \bigcup_{j = 0}^{p-1} X_i^{(j)} \right) \cup \{ y \},
\]
\[ \omega = \omega_0^{(p-1)} \omega_1^{(p-2)} \cdots \omega_{p-2}^{(1)} \omega_{p-1}^{(0)} \]
and
\[
x^{(j)} \rightarrow x^{(j+1)} \quad \text{when } x^{(j)} \in X_i^{(j)}, \; j < p - 1,
\]
\[
x^{(p-1)} \rightarrow \delta_i(x)^{(0)} y \quad \text{when } x^{(p-1)} \in X_i^{(p-1)}.
\]
Disregarding the non-commutativity of letters we may write
\[
\omega \rightarrow \delta_0(\omega_0)^{(0)} y^{\lg(\omega_0)} \omega_1^{(p-1)} \cdots \omega_{p-2}^{(1)} \omega_{p-1}^{(0)}
\]
\[ \rightarrow \delta_0(\omega_0)^{(1)} y^{\lg(\omega_0)} \delta_1(\omega_1)^{(0)} y^{\lg(\omega_1)} \cdots \omega_{p-1}^{(2)}
\]
\[ \rightarrow \ldots
\]
Thus
\[
r(n) = (r(0) + s(0) + \ldots + s(m - 1))
\]
\[ + (s(m) + \ldots + s(m + p - 1))
\]
\[ + s(m + p) + \ldots + s(n - 1)
\]
\[ = (r(0) + s(0) + \ldots + s(m - 1)) + f_G(n - m - p),
\]
when \(n \geq m + p\). It is now easy to extend \(G\) to a PDOL system \(G'\) for which \(f_{G'}(n) = r(n)\).

**Lemma 2**: Let \((r(n))\) be a \(Z\)-rational sequence. Then for any large natural number \(R\) the sequence defined by
\[
d(0) = d(1) = 1,
\]
\[
d(2n) = R^{2n} - r(2n),
\]
\[
d(2n + 1) = R^{2n} - r(2n + 1) \quad (n > 0)
\]
is a DOL sequence.

**Proof**: Let at first \((r(n))\) be a DOL sequence corresponding to the system \(G = (X, \delta, \omega)\). Construct the system \(H = (X \cup X \cup \overline{X} \cup \{ a, b \}, \delta', \Omega)\) where
\[
\Omega = a^{R^2 - r(2) - r(3)} \delta^2(\omega) \delta^3(\omega)
\]
and
\[ x \to bx, \quad x \to \lambda, \quad a \to b, \quad b \to a^{R^2}, \]
\[ \bar{x} \to a^{R^2} (1 + \lg (\delta(x))) - 21g (\delta^2 (x)) - 1g (\delta^3 (x)) \delta^2 (x) \delta^3 (x). \]

It is immediately seen that \( f_H(n) = d(n+2) \).

This implies our lemma because of the following. It is known that every \( Z \)-rational sequence is the difference of two \( N \)-rational sequences (see [9] remark 2 or [2] p. 218). Furthermore, every \( N \)-rational sequence is the difference of two DOL sequences for
\[ PM^n Q = (P + (1, \ldots, 1)) M^n Q - (1, \ldots, 1) M^n Q. \]

Hence every \( Z \)-rational sequence can be written as the difference of two DOL sequences.

**Theorem 3:** Not every increasing DOL sequence is a PDOL sequence.

**Proof:** Using lemma 2 we see that when \( R \) is a large natural number then the sequence \( (d(n)) \) where
\[
\begin{align*}
d(0) &= d(1) = 1, \\
d(2n) &= R^{2n}, \\
d(2n+1) &= R^{2n} + (\Re(3+4i)^{n+1})^2 \quad (n > 0)
\end{align*}
\]
is an increasing DOL sequence. Now
\[ \Re(3+4i)^n = \cos 2\pi n \alpha.5^n, \]
where \( \alpha \) is irrational because for every positive \( n \) \( \text{Im} (3+4i)^n < 4 \) (mod 5). The theorem of Berstel [1] then implies that the sequence \( (d(2n+1) - d(2n)) \) cannot be \( N \)-rational. Therefore \( (d(n)) \) is not a PDOL sequence.

**Note:** Let \( (d(n)) = (PM^n Q) \) be a DOL sequence. By lemma 4 below we have \( M^{m+p} \geq M^m \) for some integers \( m \) and \( p \) \((p > 0)\). But then each of the sequences
\[ (d(m+i+(n+1)p) - d(m+i+np)) \quad (i=0, \ldots, p-1) \]
is a DOL sequence and so the sequences
\[ (d(m+i+np)) \quad (i=0, \ldots, p-1) \]
are PDOL sequences. This result also appears in [5] (proof of th. 4.12).

**4. The DOL Sequences**

Let \( G = (\{ x_1, \ldots, x_k \}, \delta) \) be a DOL scheme such that for any letter \( x_i \):
\[ \lg (\delta^n(x_i)) \sim g_i A^n \quad \text{as} \quad n \to \infty \quad (g_i > 0, A \geq 1). \]
If $w \in \{x_1, \ldots, x_k\}^+$ then the number
\[g(w) = g_1(w)g_1 + \ldots + g_k(w)g_k\]
is called the growth coefficient of $w$.

Suppose we have $p$ DOL schemes $G_i = (X_i, \delta_i) \ (i = 0, \ldots, p-1)$ satisfying the condition of the above definition with a common number $A$. Introduce the infinite alphabet
\[X = \{(W_0, \ldots, W_{p-1}) \mid W_i \in X_i^+\}\]
and define
\[\delta(W_0, \ldots, W_{p-1}) = (\delta_0 W_0, \ldots, \delta_{p-1} W_{p-1})\]

Define further
\[\pi(W_0, \ldots, W_{p-1}) = (\eta(W_0), \ldots, \eta(W_{p-1}))\]
where $\eta$ means Parikh-vector. Take then a fixed element $(\omega_0, \ldots, \omega_{p-1})$ of $X$ and denote
\[Y = \left\{ (W_0, \ldots, W_{p-1}) \left| \frac{g(W_0)}{g(\omega_0)} = \ldots = \frac{g(W_{p-1})}{g(\omega_{p-1})}\right. \right\} .\]

Obviously $(W_0, \ldots, W_{p-1}) \in Y$ implies that also $\delta(W_0, \ldots, W_{p-1}) \in Y$.

**Lemma 3**: There are vectors $V_1, \ldots, V_j$ of $\pi(Y)$ such that any vector in $\pi(Y)$ is a sum of these.

**Proof**: Let $\pi(Y) \subseteq N^k$. By the definition of $Y$ there are algebraic numbers $a_{ij} \ (i = 1, \ldots, p-1; j = 1, \ldots, k)$ such that $v = (n_1, \ldots, n_k) \in Z^k$ is in $\pi(Y)$ iff
\[v \neq 0, \quad v \geq 0,\]
and
\[a_{n_1} + \ldots + a_{n_k} = 0 \quad (i = 1, \ldots, p-1).\]

Let $F$ be the additive subgroup of $Z^k$ defined by the above linear system.

We shall need the following simple lemma (see [3]):

**Lemma 4**: Any subset of $N^k$ contains only a finite number of minimal vectors (with respect to the natural componentwise ordering).

Let now $\bar{V}_1, \ldots, \bar{V}_j$ be the minimal vectors of $\pi(Y)$. If $\bar{V} \in \pi(Y)$ then it has a representation $\bar{V} = \bar{V}_i + \bar{U}$ where $\bar{U} \in N^k$. But if $\bar{U} \neq \bar{0}$ it is in $\pi(Y)$ because it belongs to $F$. Repeating this process we obtain $\bar{V}$ as a sum of the minimal vectors.

**Theorem 4**: Let $G_i = (X_i, \delta_i, \omega_i) \ (i = 0, \ldots, p-1)$ be DOL systems such that if $x_j \in X_i$ then
\[\log(\delta_i(x_j)) \sim g_{ij} A^n \quad \text{as } n \to \infty \quad (g_{ij} > 0, A \geq 1).\]
Then the sequence defined by
\[ d(np+i) = \lg(\delta_i^n(\omega_i)) \quad (n = 0, 1, \ldots, i = 0, \ldots, p-1) \]
is a DOL sequence.

**Proof:** Take \( p \) copies \( X_1^{(0)}, \ldots, X_{p-1}^{(p-1)} \) of each \( X \) and define
\[
Y = \bigcup_{j=0}^{p-1} \left\{ (W_j^{(j)}, \ldots, W_{p-1}^{(j)}) \mid W_l^{(j)} \in X_l^{(j)} \right\}, \quad \frac{g(W_0)}{g(\omega_0)} = \ldots = \frac{g(W_{p-1})}{g(\omega_{p-1})}.
\]
Let \( V_1^{(0)}, \ldots, V_j^{(0)}, \ldots, V_{j-1}^{(p-1)}, \ldots, V_{j-1}^{(p-1)} \) be elements of \( Y \) corresponding to the vectors given by lemma 3. We may say that any element of \( Y \) is a commutative product of these elements.

Introduce the DOL system
\[
G = (\{ V_1^{(0)}, \ldots, V_{j-1}^{(p-1)} \} \cup \{ y \}, \delta, \omega) = (Z \cup \{ y \}, \delta, \omega),
\]
where \( \delta \) and \( \omega \) are as follows:

\( \omega \) consists of \( (\omega_0^{(0)}, \ldots, \omega_{p-1}^{(0)}) \) written commutatively in the alphabet \( Z \) and of so many \( y \)'s that \( \lg(\omega) \) becomes equal to \( \lg(\omega_0) \);

\( y \) produces \( \lambda \);

when \( j < p-1 \) \( (W_0^{(j)}, \ldots, W_{p-1}^{(j)}) \) produces \( (W_0^{j+1}, \ldots, W_{p-1}^{j+1}) \) and so many \( y \)'s that the length of the produced word will be \( \lg(W_{j+1}) \);

\( (W_0^{(p-1)}, \ldots, W_{p-1}^{(p-1)}) \) produces \( \delta_0^{(0)}(W_0^{(0)}), \ldots, \delta_{p-1}^{(0)}(W_{p-1}^{(0)}) \) written commutatively in the alphabet \( Z \) and so many \( y \)'s that the produced word will have length \( \lg(\delta_0(W_0)) \).

Heuristically, derivations in the systems \( G \) are simulated in the components of the letters of \( Z \). With the aid of the \( p \) copies taken of the alphabets the simulation is delayed to happen only at intervals of \( p \) steps. By using the letter \( y \) the length of the word \( \delta_i+np(\omega) \) is adjusted to be equal to that of \( \delta_i^n(\omega_i) \). This is possible because the components of the letters of \( Z \) are non-empty words.

It should be clear now that \( (d(n)) \) is the growth sequence of \( G \).

Let \( G = (X, \delta) \) be a DOL scheme which gives a growth of order \( n! A^n (A \geq 1, l \geq 0) \) but does not give a growth of higher order. We divide \( X \) into classes \( \Sigma, \Sigma_0, \ldots, \Sigma_l \) as follows: the letters of \( \Sigma \) generate a growth having smaller order than \( A^n \) and the letters of \( \Sigma_i \) generate a growth of order \( n! A^n \).

It is clear that a letter of \( \Sigma_i \) cannot produce letters of \( \Sigma_{i+1} \cup \ldots \cup \Sigma_l \), it must produce a letter of \( \Sigma \) and it may produce letters of \( \Sigma \cup \Sigma_0 \cup \ldots \cup \Sigma_{i-1} \); a letter of \( \Sigma \) may produce only letters of \( \Sigma \) or \( \lambda \).

**Lemma 5:** Any letter of \( \Sigma_i (l > 0) \) generates letters of \( \Sigma_{l-1} \). If all letters of \( \Sigma \cup \Sigma_0 \cup \ldots \cup \Sigma_{l-1} \) are deleted then the resulting scheme \( H \) is such that all letters generate a growth of order \( A^n \).

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Proof: Suppose $x$ generates in $H$ a growth whose order is at least $n A^n$. Then $x$ generates in $G$ in $2^n$ steps a word whose length is at least of the order

$$n A^n n^l A^n = \left(\frac{1}{2}\right)^{l+1} (2n)^{l+1} A^{2n}.$$  

This shows that the growth of $x$ in $H$ has the order $A^n$ at most.

Assume that $x \in \Sigma_l (l > 0)$ never generates a letter of $\Sigma_{l-1}$. By the above $\delta^n (x)$ contains $O (A^n)$ letters of $\Sigma \cup \Sigma_0 \cup \ldots \cup \Sigma_{l-2}$ directly produced by letters of $\Sigma_l$. But

$$\sum_{n=0}^{N} A^n (N-n)^{l-2} A^{N-n} = A^N \sum_{n=0}^{N} n^{l-2} = o (N^l A^N)$$

and so $x$ cannot generate a growth of order $n^l A^n$. Hence $x$ must generate letters of $\Sigma_{l-1}$.

Assume further that the growth of $x$ in $H$ is majorized by $a^n (a < A)$. Because

$$\sum_{n=0}^{N} a^n (N-n)^{l-1} A^{N-n} = a^N \sum_{n=0}^{N} n^{l-1} (A/a)^n = O (N^{l-1} A^N)$$

we have a contradiction as above. Thus the growth of $x$ in $H$ has the order $A^n$ when $l > 0$.

Suppose $x \in \Sigma_0$ and the growth of $x$ in $H$ as well as the growth of any letter of $\Sigma$ in $G$ is majorized by $a^n (a < A)$. Because

$$\sum_{n=0}^{N} a^n A^{N-n} = o (A^N)$$

we see that the above result is true also when $l = 0$.

Theorem 5: Let $(r (n))$ be an $N$-rational sequence such that $r (n) \neq 0$ for every $n$ and the quotient $r (n+1)/r (n)$ remains bounded. Then $(r (n))$ is a DOL sequence.

Proof: We know that there are numbers $m$ and $p$ and DOL sequences $(d_0 (n)), \ldots, (d_{p-1} (n))$ such that

$$r(m+l+np) = d_i(n).$$

By our assumption all these sequences have the same order of growth. We may suppose that it is of the form $n^l A^n (A > 1)$ for the polynomial case is covered by a theorem of Ruohonen [7].

Let $G_i = (X_i, \delta_i, \omega_i)$ be a DOL system corresponding to the sequence $(d_i (n))$. Write $X_i = \Sigma_i \cup \Sigma_{i0} \cup \ldots \cup \Sigma_{ij}$ as before and denote by $H_{ij}$ the DOL schema obtained from $(X_i, \delta_i)$ by deleting all letters except those of $\Sigma_{ij}$.

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Suppose $W \in \Sigma_{i0}^+$ and $w \in \Sigma_i^*$. If $W$ generates $V_v (V \in \Sigma_{i0}^+, v \in \Sigma_i^*)$ and $w$ generates $u$ in $k$ steps then there are constants $N, M$ and $L$ (independent of $k$ and $i$) such that

$$\log(V) \geq NA^k \log(W),$$
$$\log(v) \leq MA^k \log(W),$$
$$\log(u) \leq La^k \log(w) \quad (a < A).$$

We now see that if $\log(w)/\log(W) \leq \alpha M/N (\alpha \geq 2, \alpha M/N \text{ integer})$ and if $k$ is so large that $(L/N) (a/A)^k \leq 1/2$ then

$$\log(w)/\log(V) \leq M/N + (L/N)(a/A)^k (\log(w)/\log(W)) \leq M/N + (1-1/\alpha) \alpha M/N = \alpha M/N,$$

too.

By taking a multiple of $p$, if necessary, we may suppose that the following three conditions hold:

(i) any growth in $H_{ij} (i = 0, \ldots, p-1; j = 0, \ldots, l)$ is asymptotically equal to constant times $A^n$;

(ii) any letter of $\Sigma_{ij} (i = 0, \ldots, p-1; j = 0, \ldots, l)$ produces in a step letters of all the alphabets $\Sigma_{i,j-1}, \ldots, \Sigma_{i,0}$;

(iii) the equation (1) holds with $k = 1$.

Moreover, by increasing $m$ we obtain the following situation:

(iv) the axiom of $G_i$ contains letters of all the alphabets $\Sigma_{i,0}, \ldots, \Sigma_{il}$.

We are now ready to give an induction proof showing that the sequence $(s(n)) = (r(n-m))$ is a DOL sequence. This immediately implies our theorem.

If $l = 0$ we at first neglect all letters of the $\Sigma_i$'s and construct a system just as in the proof of theorem 4. Then we take $p$ copies $x^{(0)}, \ldots, x^{(p-1)}$ of each neglected letter $x$ and join these copies to the components of the letters of $Z$ so that the original systems $G_i$ become simulated. Condition (iii) assures that this can be done. At the same time we add $y$'s so that the right lengths are obtained.

When taking the induction step we at first delete all letters of the alphabets $\Sigma_i \cup \Sigma_{i0} \cup \ldots \cup \Sigma_{i,l-1}$ and construct a system according to theorem 4. By conditions (ii) and (iv) the letters of this system as well as the $p$-tuple $(\omega_0, \ldots, \omega_{p-1})$ give axioms for systems whose existence is guaranteed by the induction hypothesis.

Example: Let

$$G_0 = (\{A, B, C\}, \delta_0, AB),$$

where

$$A \to A^4 B, \quad B \to B^4 b, \quad b \to b$$

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and

\[ G_1 = (\{ C, D, E, F \}, \delta_1, C D), \]

where

\[ C \rightarrow C^4 D, \quad D \rightarrow D^2 E^3, \quad E \rightarrow E^2 F^4, \quad F \rightarrow DF. \]

The common order of growth is \( n \cdot 4^n \) and

\[ \Sigma_0 = \{ b \}, \quad \Sigma_{00} = \{ B \}, \quad \Sigma_{01} = \{ A \}, \]

\[ \Sigma_1 = \emptyset, \quad \Sigma_{10} = \{ C, D, E, F \}, \quad \Sigma_{11} = \{ C \}. \]

The procedures described in the preceding proof yield e.g. the following system: The axiom is

\[ (A^{(0)}, C^{(0)})(B^{(0)}, D^{(0)}) \]

and the productions are

\[ (A^{(0)}, C^{(0)}) \rightarrow (A^{(1)}, C^{(1)}), \]
\[ (A^{(1)}, C^{(1)}) \rightarrow (A^{(0)}, C^{(0)})^4 (B^{(0)}, D^{(0)}), \]
\[ (B^{(0)}, D^{(0)}) \rightarrow (B^{(1)}, D^{(1)}), \]
\[ (B^{(1)}, D^{(1)}) \rightarrow (B^{(0)} b^{(0)}, D^{(0)})(B^{(0)}, D^{(0)})(B^{(0)} b^{(0)}, E^{(0)} y^2)(B^{(0)}, E^{(0)}) y^2, \]
\[ (B^{(0)}, E^{(0)}) \rightarrow (B^{(1)}, E^{(1)} y^2), \]
\[ (B^{(1)}, E^{(1)}) \rightarrow (B^{(0)} b^{(0)}, E^{(0)} y^2)(B^{(0)}, F^{(0)} y^2), \]
\[ (B^{(0)}, F^{(0)}) \rightarrow (B^{(1)}, F^{(1)} y^2), \]
\[ (B^{(1)}, F^{(1)}) \rightarrow (B^{(0)} b^{(0)}, F^{(0)} y^2)(B^{(0)}, D^{(0)} y^3), \]
\[ (B^{(0)} b^{(0)}, D^{(0)}) \rightarrow (B^{(1)} b^{(1)}, D^{(1)}), \]
\[ (B^{(1)} b^{(1)}, D^{(1)}) \rightarrow (B^{(0)} b^{(0)}, D^{(0)} y^2)(B^{(0)} b^{(0)}, E^{(0)} y^3), \]
\[ (B^{(0)} b^{(0)}, E^{(0)}) \rightarrow (B^{(1)} b^{(1)}, E^{(1)} y^2), \]
\[ (B^{(1)} b^{(1)}, E^{(1)}) \rightarrow (B^{(0)} b^{(0)}, E^{(0)} y^2)(B^{(0)} b^{(0)}, F^{(0)} y^3)(B^{(0)}, F^{(0)} y^3) y^5, \]
\[ (B^{(0)} b^{(0)}, F^{(0)}) \rightarrow (B^{(1)} b^{(1)}, F^{(1)} y^2), \]
\[ (B^{(1)} b^{(1)}, F^{(1)}) \rightarrow (B^{(0)} b^{(0)}, F^{(0)} y^2)(B^{(0)} b^{(0)}, D^{(0)})(B^{(0)}, D^{(0)} y^2) y^2, \]

\[ y \rightarrow \lambda. \]

Note: Let \( d(n) \) be a DOL sequence such that the rational function \( \sum d(n) x^n \) is not a polynomial. Then it is easy to see that the growth order of \( d(n) \) is \( n^l A^n \) where \( 1/A \) is the smallest positive pole of \( \sum d(n) x^n \) and \( l+1 \)
is its multiplicity. This enables us to effectively compare the growth orders of two DOL sequences; we describe the method briefly in general form.

Given integer polynomials \( q_1(x), \ldots, q_p(x) \) we can (using symmetric polynomials) construct a polynomial \( Q(x) \) such that any difference of two zeros of \( q(x) = q_1(x) \ldots q_p(x) \) is a zero of \( Q(x) \). Thus we can give a positive number \( \gamma \) such that if \( z_1 \) and \( z_2 \) are zeros of \( q(x) \) then either \( z_1 = z_2 \) or \( |z_1 - z_2| > \gamma \).

The polynomial

\[
Q_i(x) = \frac{q_i(x)}{\text{g.c.d.}(q_1(x), q_i(x))}
\]

has simple zeros which are the same as those of \( q_i(x) \). Therefore we can compare the real roots of the polynomials \( q_i(x) \) by examining the sign changes of the polynomials \( Q_i(x) \).

Because \( x \) is a \( k \)-fold zero of \( q_i(x) \) iff it is a zero of \( q_1(x), q_i(x), \ldots, q^{(k-1)}(x) \) but not a zero of \( q^{(k)}(x) \) it is possible to determine the multiplicity of any real zero of \( q_i(x) \).

An \( N \)-rational sequence \( (r(n)) \) is by theorem 5 a DOL sequence iff one of the following conditions holds:

(i) there is a natural number \( L \) such that \( r(n) > 0 \) when \( n \leq L \) and \( r(n) = 0 \) when \( n > L \);

(ii) every \( r(n) \) is positive and the DOL sequences given by theorem 1 have the same growth order.

Hence it is possible to decide whether or not a given \( N \)-rational sequence is a DOL sequence.

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REFERENCES


décembre 1976.


