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A SYSTEM OF TWO DIOPHANTINE INEQUALITIES WITH PRIMES

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Abstract. Let $k \geq 7$ be fixed, and $v = \frac{55k+556}{26k+672}, 1 < d < c < v$, $1 < \mu < \nu < k^{1-d/c}$ and $\mu \leq N_2/N_1^{d/c} \leq \nu$. Suppose that $N_1', N_2' > 0$ are real numbers, and that N_1, N_2 are real numbers satisfying $N_1 > N_1'$ and $N_2 > N_2'$. Then we prove the system of two Diophantine inequalities

$$|p_1^c + \cdots + p_k^c - N_1| < N_1^{-(1/c)(v-c)} \log^{195} N_1,$$
$$|p_1^d + \cdots + p_k^d - N_2| < N_2^{-(1/d)(v-d)} \log^{195} N_2$$

has prime solutions $p_1, \cdots, p_k$.

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1. Introduction

The Waring–Goldbach problem is a famous and still incompletely resolved question in number theory. It concerns whether, for a given exponent $k \geq 2$, there exists a minimal $s(k)$ such that every sufficiently large integer N satisfying the necessary congruence conditions, can be expressed as the sum of $s(k)$ k-th powers of prime numbers.

The problem of solving Diophantine inequalities in primes is a natural generalization of the Waring–Goldbach problem, Piatetski–Shapiro [1] first considered the Diophantine inequality

$$|p_1^c + p_2^c + \cdots + p_t^c - N| < \varepsilon, \tag{1.1}$$

where $c > 1$ is a non-integer, $\varepsilon > 0$ is arbitrarily small and N is a large real number. Let $H(c)$ denote the least integer for which the inequality (1.1) is solvable in primes $p_1, p_2, \cdots, p_t$. He established that

$$\overline{\lim_{c \to \infty}} \frac{H(c)}{c \log c} \leq 4. \tag{1.2}$$

Additionally, he demonstrated that $H(c) \leq 5$ holds for $1 < c < \frac{3}{2}$.

Keywords and phrases: Diophantine inequality, prime, exponential sum, additive problems.

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The upper bound of c has been improved by scholars such as Zhai and Cao [2], Liu and Shi [3]. The best result so far is 378/181, which is proved by Baker [4]. In addition, many scholars have studied a lot of this type of similar problems (see [12–14], [15], [16], [18–21]).

In 2021, Yan and Zhang [5] extended it to a more general case. Let $t \geq 5$ be a fixed integer, the inequality (1.1) with $\varepsilon = N^{-(9/10c)(\delta-c)}$ has prime solutions $p_1, \cdots, p_t$ for $2 < c < \delta$, where $\delta = \frac{294-210t}{123-97t}$.

A pioneering study of a system of two Diophantine inequalities was conducted by Tolev [6]. Let c, d, μ, ν are arbitrary numbers. He considered

$$\begin{aligned} &|p_1^c + \cdots + p_k^c - N_1| < \varepsilon_1, \\ &|p_1^d + \cdots + p_k^d - N_2| < \varepsilon_2, \end{aligned} \tag{1.3}$$

where ε_1 and ε_2 are sufficiently small functions depending on N_1 and N_2 respectively, and N_1, N_2 satisfy $N_1 > N_1', N_2 > N_2'$ and $\mu \leq N_2/N_1^{d/c} \leq \nu$. When $k = 5$, he proved the system (1.3) holds for

$$\varepsilon_1 = N_1^{-\frac{35-34c}{34c}} \log^{12} N_1, \quad \varepsilon_2 = N_2^{-\frac{35-34d}{34d}} \log^{12} N_2$$

with

$$1 < d < c < \frac{35}{34}, \quad 1 < \mu < \nu < 5^{1-\frac{d}{c}}.$$

Later, Zhai [17] improved this result. In 2002, Zhai and Cao [7] obtained the best results so far, and proved that the system holds for

$$\varepsilon_1 = N_1^{-\frac{27-26c}{26c}} \log^{100} N_1, \quad \varepsilon_2 = N_2^{-\frac{27-26d}{26d}} \log^{100} N_2$$

with $1 < d < c < \frac{27}{26}$.

In 2021, Han, Liu and Zhang [8] considered the ststem (1.3) with $k = 6$. They proved the system (1.3) with

$$\begin{aligned} \varepsilon_1 &= N_1^{-\frac{128-119c}{119c}} \log^{109} N_1, \\ \varepsilon_2 &= N_2^{-\frac{128-119d}{119d}} \log^{109} N_2 \end{aligned}$$

has prime solutions $p_1, p_2, \cdots, p_6$ for $1 < d < c < \frac{128}{119}$ and $1 < \mu < \nu < 6^{1-\frac{d}{c}}$.

Inspired by Yan and Zhang [5], we consider the system (1.3) with $k \geq 7$. Suppose $c > 1$, $d > 1$ are distinct numbers and $\varepsilon_1, \varepsilon_2$ satisfy

$$\begin{aligned} &\varepsilon_1 \to 0, \text{ as } N_1 \to \infty, \\ &\varepsilon_2 \to 0, \text{ as } N_2 \to \infty. \end{aligned}$$

Since the inequality

$$(x_1^c + \cdots + x_k^c)^{\frac{d}{c}} \leq x_1^d + \cdots + x_k^d \leq k^{1-\frac{d}{c}} (x_1^c + \cdots + x_k^c)^{\frac{d}{c}},$$

a restriction on the orders of N_1 and N_2 should be imposed. For each positive $x_1, \cdots, x_k$ as long as $1 < d < c$, the above inquality holds. Our result is as follows.

Theorem 1.1. *Assume that $k \geq 7$ is a fixed integer. There exist real numbers c, d, μ, ν satisfying the following inequalities*

$$1 < d < c < v = \frac{55k + 556}{26k + 672}, \tag{1.4}$$

$$1 < \mu < \nu < k^{1-\frac{d}{c}}. \tag{1.5}$$

Let $N_1', N_2' > 0$ be real numbers that rely on c, d, μ, ν. If real numbers $N_1 > N_1', N_2 > N_2'$ and satisfy

$$\mu \le \frac{N_2}{N_1^{\frac{d}{c}}} \le \nu, \tag{1.6}$$

then the system (1.3) *with*

$$\varepsilon_1 = N_1^{-(1/c)(v-c)} \log^{195} N_1,$$
$$\varepsilon_2 = N_2^{-(1/d)(v-d)} \log^{195} N_2$$

is solvable in primes $p_1, \cdots, p_k$.

Remark. The results of this paper also hold for $k = 5$ and $k = 6$, but they do not reach the best known results in these cases.

2. Notion and an outline of the proof

In this paper, the letter p always stands for a prime whether or not it has a subscript. Let N_1, N_2 be sufficiently large numbers. Let $\eta > 0$ be a small number that relies on c and d. For an integer $k \ge 7$, define

$$\begin{aligned}
&X = N_1^{1/c}, \quad v = \frac{55k+556}{26k+672},\\
&\tau_1 = X^{3/4-c-\eta}, \quad \tau_2 = X^{3/4-d-\eta},\\
&\varepsilon_1 = X^{-(v-c)} \log^{195} X, \quad \varepsilon_2 = X^{-(v-d)} \log^{195} X,\\
&K_1 = X^{(v-c)} \log^{-194} X, \quad K_2 = X^{(v-d)} \log^{-194} X,\\
&e(t) = \mathrm{e}^{2\pi i t}, \quad \psi(t) = \mathrm{e}^{-\pi t^2}, \quad \psi_\delta(t) = \delta\psi(\delta t).
\end{aligned}$$

As usual, $\chi(t)$ stands for the characteristic function defined on $[-1, 1]$. Let $\rho = \beta' + i\gamma'$ be a non-trivial zero of the Riemann zeta function $\zeta(s)$. $\Lambda(n)$ is the von Mangoldt function and $d(n)$ is the divisor function.

In addition, we prepare an outline of the proof for the reader's convenience.

First, we write

$$G = \sum_{\lambda X < p_1, \cdots, p_k \le X} (\log p_1) \cdots (\log p_k) \chi\left(\frac{p_1^c + \cdots + p_k^c - N_1}{\varepsilon_1 \log X}\right) \chi\left(\frac{p_1^d + \cdots + p_k^d - N_2}{\varepsilon_2 \log X}\right), \tag{2.1}$$

where $\lambda > 0$ is small enough depending on c, d, μ, ν (the specific definition can be found in Lem. 4.4). Set

$$E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P^k(x, y) e(-N_1 x) \psi_{\varepsilon_1}(x) e(-N_2 y) \psi_{\varepsilon_2}(y) \mathrm{d}x \mathrm{d}y, \tag{2.2}$$

where

$$P(x, y) = \sum_{\lambda X < p \le X} e(xp^c + yp^d) \log p. \tag{2.3}$$

Next, the plane can be divided into the following three parts:

$$\begin{aligned} D_1 &= \left\{(x,y) : \max\left(\frac{|x|}{\tau_1}, |\frac{|y|}{\tau_2}\right) < 1\right\}, \\ D_2 &= \left\{(x,y) : \max\left(\frac{|x|}{\tau_1}, \frac{|y|}{\tau_2}\right) \geq 1, \max\left(\frac{|x|}{K_1}, \frac{|y|}{K_2}\right) \leq 1\right\}, \\ D_3 &= \left\{(x,y) : \max\left(\frac{|x|}{K_1}, \frac{|y|}{K_2}\right) > 1\right\}. \end{aligned}$$

Accordingly, we divide the integral E into three parts:

$$\begin{aligned} E &= (\iint\limits_{D_1} + \iint\limits_{D_2} + \iint\limits_{D_3})P^k(x,y)e(-N_1x)\psi_{\varepsilon_1}(x)e(-N_2y)\psi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y \\ &:= E_1 + E_2 + E_3. \end{aligned} \tag{2.4}$$

When X tends to infinity, it is expected that G tends to infinity. Then Theorem 1.1 holds. To prove that G tends to infinity as E tends to infinity, the following Lemma 3.3 suffices.

Hence, the main work of our result turns to consider the following inequalities:

$$|E_1| \gg \varepsilon_1\varepsilon_2 X^{k-c-d},$$

$$|E_2| \ll \frac{\varepsilon_1\varepsilon_2 X^{k-c-d}}{\log X},$$

$$|E_3| \ll 1.$$

In Section 4 and Section 5, we will give estimates for E_1 and E_2. The complete proof will be given in Section 6.

3. Auxiliary lemmas

In preparation for proving the theorem, we first establish some preliminary lemmas.

Lemma 3.1. *The function $\psi(t) = \mathrm{e}^{-\pi t^2}$ satisfies:*

$$\psi(x) = \int_{-\infty}^{\infty} \psi(t)e(-xt)dt, \tag{3.1}$$

$$\chi\left(\frac{t}{\varrho}\right) \geq \psi(t) - \psi(\varrho) \text{ for } \varrho > 0, \tag{3.2}$$

$$\psi(t) \geq \psi(1) \text{ for } |t| \leq 1.$$

Proof. The proof of this lemma can be found in Tolev [6]. □

Lemma 3.2. *For real number $M > 1$, f is a smooth real function defined on $[M, 2M]$. Suppose that $B = B(f) > 0$ depending on i is a constant such that*

$$BM^{1-i} \ll_i |f^i(x)| \ll_i BM^{1-i},$$

where $x \sim B$ and $i \in \mathbb{N}$.

For each exponent pair (λ, κ) satisfying

$$0 \le \lambda \le \frac{1}{2} \le \kappa < 1,$$

we have

$$\sum_{M<n\le M+a} e(f(n)) \ll B^{\lambda} M^{\kappa},$$

where $1 < a \le M$.

Proof. This lemma appears in Tolev [9]. □

Lemma 3.3. *Let G and E be defined by (2.1) and (2.2), respectively. The following inequality*

$$G \ge E + O(1)$$

holds.

Proof. By (3.2) in Lemma 3.1, the inequality

$$\chi\left(\frac{p_1^c + \cdots + p_k^c - N_1}{\varepsilon_1 \log X}\right) \ge \psi\left(\frac{p_1^c + \cdots + p_k^c - N_1}{\varepsilon_1}\right) - \psi(\log X)$$

holds. By (3.1), we have the equality

$$\begin{aligned}
\psi\left(\frac{p_1^c + \cdots + p_k^c - N_1}{\varepsilon_1}\right) &= \psi\left(\frac{N_1 - (p_1^c + \cdots + p_k^c)}{\varepsilon_1}\right) \\
&= \int_{-\infty}^{\infty} \psi(x) e\left(-\frac{N_1 - (p_1^c + \cdots + p_k^c)}{\varepsilon_1} x\right) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \varepsilon_1 \psi(\varepsilon_1 x) e\left(-(N_1 - (p_1^c + \cdots + p_k^c))x\right) \mathrm{d}x \\
&= \int_{-\infty}^{\infty} \psi_{\varepsilon_1}(x) e(-N_1 x) e(x(p_1^c + \cdots + p_k^c)) \mathrm{d}x,
\end{aligned}$$

where we replace x with $\varepsilon_1 x$. An analogous argument yields

$$\chi\left(\frac{p_1^d + \cdots + p_k^d - N_2}{\varepsilon_2 \log X}\right) \ge \psi\left(\frac{p_1^d + \cdots + p_k^d - N_2}{\varepsilon_2}\right) - \psi(\log X)$$

and

$$\psi\left(\frac{p_1^d + \cdots + p_k^d - N_2}{\varepsilon_2}\right) = \int_{-\infty}^{\infty} e(y(p_1^d + \cdots + p_k^d)) e(-yN_2) \psi_{\varepsilon_2}(y) \mathrm{d}y.$$

Then

$$
\begin{aligned}
G \geq & \sum_{\lambda X < p_1, \cdots, p_k \leq X} (\log p_1) \cdots (\log p_k) \psi \left(\frac{p_1^c + \cdots + p_k^c - N_1}{\varepsilon_1} \right) \\
& \times \psi \left(\frac{p_1^d + \cdots + p_k^d - N_2}{\varepsilon_2} \right) + O(1) \\
\geq & \sum_{\lambda X < p_1, \cdots, p_k \leq X} (\log p_1) \cdots (\log p_k) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e(x(p_1^c + \cdots + p_k^c) \\
& + y(p_1^d + \cdots + p_k^d)) e(-xN_1 - yN_2) \psi_{\varepsilon_1}(x) \psi_{\varepsilon_2}(y) \mathrm{d}x \mathrm{d}y + O(1) \\
\geq & E + O(1).
\end{aligned}
$$

□

Lemma 3.4. *For $t \in [a, b]$, let $Q(t), U(t)$ denote real functions satisfying $|Q(t)| \leq \mathcal{M}$ and $\frac{Q(t)}{U(t)}$ monotonic. Set*

$$
\mathcal{I} = \int_a^b Q(t) e(U(t)) \mathrm{d}t.
$$

Let $m > 0$. If $U'(t) \geq m$ or $U'(t) \leq m$, then

$$
|\mathcal{I}| \ll \mathcal{M}/m
$$

holds for each $t \in [a, b]$.
Suppose that $U''(t) \geq m$, then

$$
|\mathcal{I}| \ll \mathcal{M}/\sqrt{m}.
$$

Proof. This result is due to Ivić [9], pp. 56–57. □

4. The estimation of the integral E_1

In this section, we will estimate the following bound of E_1:

Proposition 4.1. *From the integral E_1 in (2.4), we obtain*

$$
|E_1| \gg \varepsilon_1 \varepsilon_2 X^{k-c-d}.
$$

Denote $T = X^{3/4+c/4v-\eta}$. Put

$$
I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H^k(x, y) e(-N_1 x) \psi_{\varepsilon_1}(x) e(-N_2 y) \psi_{\varepsilon_2}(y) \mathrm{d}x \mathrm{d}y \tag{4.1}
$$

and

$$
I_1 = \iint\limits_{D_1} H^k(x, y) e(-N_1 x) \psi_{\varepsilon_1}(x) e(-N_2 y) \psi_{\varepsilon_2}(y) \mathrm{d}x \mathrm{d}y, \tag{4.2}
$$

where

$$H(x,y) = \int_{\lambda X}^{X} e(xt^c + yt^d)\mathrm{d}t. \tag{4.3}$$

In the remaining space, we approximate I_1 by I, then use I_1 to approximate E_1, thereby deriving the bound for E_1.

Lemmas 7, 8, and 9 in Tolve [6] provide, respectively, the relationship between $P(x,y)$ and $H(x,y)$, an upper bound estimate for $P(x,y)$, and an upper bound estimate for $H(x,y)$.

Lemma 4.2. *Suppose that $(x,y) \in D_1$, then*

$$P(x,y) = H(x,y) + O(X\mathrm{e}^{-(\log X)^{1/5}}). \tag{4.4}$$

By the definition of (2.3) and (4.3), the following inequalities hold:

$$\iint\limits_{D_1} |P(x,y)|^4 \psi_{\varepsilon_1}(x)\psi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y \ll \varepsilon_1\varepsilon_2 X^{4-c-d}\log^8 X, \tag{4.5}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |H(x,y)|^4 \psi_{\varepsilon_1}(x)\psi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y \ll \varepsilon_1\varepsilon_2 X^{4-c-d}\log^4 X. \tag{4.6}$$

Lemma 4.3. *As noted in (4.1) and (4.2) of I and I_1, respectively, then the following inequality*

$$|I - I_1| \ll \frac{\varepsilon_1\varepsilon_2 X^{k-c-d}}{\log X}$$

holds.

Proof. Using (4.6) in Lemma 4.2, we get

$$\begin{aligned} |I - I_1| &\ll \iint\limits_{\mathbb{R}^2\setminus D_1} |H(x,y)|^k \psi_{\varepsilon_1}(x)\psi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y \\ &\ll \max_{\mathbb{R}^2\setminus D_1} |H(x,y)|^{k-4} \iint\limits_{\mathbb{R}^2} |H(x,y)|^4 \psi_{\varepsilon_1}(x)\psi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y \\ &\ll \varepsilon_1\varepsilon_2 X^{4-c-d}\log^4 X \max_{\mathbb{R}^2\setminus D_1} |H(x,y)|^{k-4}. \end{aligned} \tag{4.7}$$

Applying Lemma 3.4 and letting $Q(t) = 1$, $U(t) = xt^c + yt^d$, we obtain

$$\max_{\mathbb{R}^2\setminus \Omega_1} |H(x,y)| \ll X^{5/8+\eta/2}, \tag{4.8}$$

since

$$U''(x) = c(c-1)xt^{c-2} + d(d-1)yt^{d-2} \geq (c-1)(d-1)X^{-5/4-\eta}.$$

By (4.7) and (4.8), we get

$$|I - I_1| \ll \frac{\varepsilon_1 \varepsilon_2 X^{k-c-d}}{\log X}.$$

□

Lemma 4.4. *Let V be the volume of a domain in $\mathbb{R}^k$. Let σ_1, σ_2 be sufficiently small and $\delta \in [\mu, \nu]$. For given parameters c, d, μ, ν, one can find a constant $\lambda > 0$ such that if*

$$t_1, \cdots, t_k > \lambda,\ |t_1^c + \cdots + t_k^c - 1| < \sigma_1,\ |t_1^d + \cdots + t_k^d - \delta| < \sigma_2,$$

then

$$V \gg \sigma_1 \sigma_2.$$

Proof. This lemma bears a close resemblance to Lemma 1 established by Tolve [6]. The volume of V can be written as

$$V = \int \cdots \int_{\substack{t_1, \ldots, t_k > \lambda \\ |t_1^c + \cdots + t_k^c - 1| \le \sigma_1 \\ |t_1^d + \cdots + t_k^d - 1| \le \sigma_2}} 1 \, dt_1 \cdots dt_k.$$

By fixing $t_1, \cdots, t_{k-1}$, we obtain the range of t_k:

$$(1 - S_{k-1}^c - \sigma_1)^{1/c} \le t_k \le (1 - S_{k-1}^c + \sigma_1)^{1/c},$$

$$(\delta - S_{k-1}^d - \sigma_2)^{1/d} \le t_k \le (\delta - S_{k-1}^d + \sigma_2)^{1/d},$$

where $S_{k-1}^c = \sum_{i=1}^{k-1} t_i^c$ and $S_{k-1}^d = \sum_{i=1}^{k-1} t_i^d$. Since σ_1, σ_2 are sufficiently small, we can adjust λ such that the two inequalities above have an intersection.

□

Lemma 4.5. *By the definition (4.1) of I, then*

$$I \gg \varepsilon_1 \varepsilon_2 X^{k-c-d}.$$

Proof. By a proof similar to that of Lemma 3.3, it can be obtained that

$$\begin{aligned} I = & \int \cdots \int_{\lambda X < t_1, \cdots, t_k \le X} \int_{-\infty}^{\infty} e(x(t_1^c + \cdots + t_k^c - N_1)) \psi_{\varepsilon_1}(x) \mathrm{d}x \\ & \times \int_{-\infty}^{\infty} e(y(t_1^d + \cdots + t_k^d - N_2)) \psi_{\varepsilon_2}(y) \mathrm{d}y dt_1 \cdots dt_k \\ = & \int \cdots \int_{\lambda X < t_1, \cdots, t_k \le X} \psi\left(\frac{t_1^c + \cdots + t_k^c - N_1}{\varepsilon_1}\right) \psi\left(\frac{t_1^d + \cdots + t_k^d - N_2}{\varepsilon_2}\right) dt_1 \cdots dt_k. \end{aligned} \tag{4.9}$$

We divide the integration region into two parts:

$$Z_1 = \left\{ \lambda X < t_1, \cdots, t_k \le X : \left| \frac{t_1^c + \cdots + t_k^c - N_1}{\varepsilon_1} \right| \le 1 \text{ and } \left| \frac{t_1^d + \cdots + t_k^d - N_2}{\varepsilon_2} \right| \le 1 \right\},$$

$$Z_2 = \left\{ \lambda X < t_1, \cdots, t_k \le X : \left| \frac{t_1^c + \cdots + t_k^c - N_1}{\varepsilon_1} \right| > 1 \text{ or } \left| \frac{t_1^d + \cdots + t_k^d - N_2}{\varepsilon_2} \right| > 1 \right\}.$$

Let

$$\Delta = \psi \left(\frac{t_1^c + \cdots + t_k^c - N_1}{\varepsilon_1} \right) \psi \left(\frac{t_1^d + \cdots + t_k^d - N_2}{\varepsilon_2} \right).$$

In (4.9), we obtain $\int\limits_{Z_2} \cdots \int \Delta dt_1 \cdots dt_k \ll 1$, since the integral converges on Z_2. Substituting the result into the integral I yields

$$I = \int\limits_{Z_1} \cdots \int \Delta dt_1 \cdots dt_6 + O(1) = X^k \int\limits_{Z_1'} \cdots \int \Delta d \left(\frac{t_1}{X} \right) \cdots d \left(\frac{t_k}{X} \right) + O(1),$$

where

$$Z_1' = \left\{ \lambda < \frac{t_1}{X}, \cdots, \frac{t_k}{X} : \left| \left(\frac{t_1}{X} \right)^c + \cdots + \left(\frac{t_k}{X} \right)^c - 1 \right| \le \frac{\varepsilon_1}{X^c} \text{ and } \left| \left(\frac{t_1}{X} \right)^d + \cdots + \left(\frac{t_k}{X} \right)^d - \frac{N_2}{X^d} \right| \le \frac{\varepsilon_2}{X^d} \right\}.$$

By using (1.6) and Lemma 4.4, we can get

$$I \gg \varepsilon_1 \varepsilon_2 X^{k-c-d},$$

which obtains this lemma. □

Proof of Proposition 4.1. It follows from the definitions of E_1 and I_1 that

$$\begin{aligned} |E_1 - I_1| &\ll \iint\limits_{D_1} |P^k(x,y) - H^k(x,y)| \psi_{\varepsilon_1}(x) \psi_{\varepsilon_2}(y) \mathrm{d}x \mathrm{d}y \\ &\ll \max_{D_1} |P(x,y) - H(x,y)|^{k-4} \iint\limits_{D_1} (|P(x,y)|^4 + |H(x,y)|^4) \psi_{\varepsilon_1}(x) \psi_{\varepsilon_2}(y) \mathrm{d}x \mathrm{d}y. \end{aligned} \tag{4.10}$$

For the first part, we use (4.4) in Lemma 4.2 to get

$$P(x,y) - H(x,y) \ll X \mathrm{e}^{-(\log X)^{1/5}}.$$

For the integral, we apply (4.5) and (4.6) in Lemma 4.2 to obtain

$$\iint\limits_{D_1} (|P(x,y)|^4 + |H(x,y)|^4) \psi_{\varepsilon_1}(x) \psi_{\varepsilon_2}(y) \mathrm{d}x \mathrm{d}y \ll \varepsilon_1 \varepsilon_2 X^{4-c-d} \log^8 X.$$

Substituting the above two estimates into (4.10) yields

$$|E_1 - I_1| \ll \frac{\varepsilon_1\varepsilon_2 X^{k-c-d}}{\log X}.$$

By Lemma 4.3, we deduce

$$|E_1 - I| \ll \frac{\varepsilon_1\varepsilon_2 X^{k-c-d}}{\log X}.$$

It is clear that $|E_1| \gg \varepsilon_1\varepsilon_2 X^{k-c-d}$ from Lemma 4.5. □

5. The estimation for the integral E_2

Throughout this section, we will use exponential sum estimates to prove the following bound of E_2:

Proposition 5.1. *For the integral E_2 in (2.4), we have*

$$|E_2| \ll \frac{\varepsilon_1\varepsilon_2 X^{k-c-d}}{\log X}.$$

The following lemmas are necessary tools for proving Proposition 5.1.

Lemma 5.2. *Consider arbitrary complex sequences a_k and b_l. Let $(x,y) \in D_2$, $X^{1/4} < L < L_1 \le 2L$, $X^{1/4} < R \le X^{1/2}$, $LR \le X$.*
For

$$V = \sum_{X^{1/4}<k\le R} \sum_{L<l\le L_1} a_k b_l e((kl)^c x + (kl)^d y),$$

we have the estimate

$$|V| \ll (\mathcal{A}\mathcal{B})^{1/2} X^{(26v+139)/388} \log^{-13} X,$$

where

$$\mathcal{A} = \sum_{X^{1/4}<k\le R} |a_k|^2, \quad \mathcal{B} = \sum_{L<l\le L_1} |b_l|^2.$$

Proof. For the case $x \in [\tau_1, K_1], y \in [0, K_2]$. We define

$$R_i = \begin{cases} X^{1/4}, & i = 0, \\ \min(R_i + s, R), & i \in [1, Q-1] \\ R, & i = Q. \end{cases}$$

where $s \in [1, R]$, $Q \ll \frac{R}{s}$. Then the summation V can be rewritten as

$$\begin{aligned} V &= \sum_{L<l\le L_1} \sum_{1\le i\le Q} \sum_{R_{i-1}<k\le R_i} a_k b_l e((kl)^c x + (kl)^d y) \\ &\le \sum_{L<l\le L_1} |b_l| \left| \sum_{1\le i\le Q} \sum_{R_{i-1}<k\le R_i} a_k e((kl)^c x + (kl)^d y) \right|. \end{aligned}$$

Applying Cauchy's inequality, we get

$$\begin{aligned} |V|^2 &\le \mathcal{B} \sum_{L<l\le L_1} \left| \sum_{1\le i\le Q} \sum_{R_{i-1}<k\le R_i} a_k b_l e((kl)^c x + (kl)^d y) \right|^2 \\ &\le \mathcal{B}Q \sum_{L<l\le L_1} \sum_{1\le i\le Q} \sum_{R_{i-1}<k_1,k_2\le R_i} a_{k_1} \bar{a}_{k_2} e((k_1^c - k_2^c) l^c x + (k_1^d - k_2^d) l^d y) \\ &\ll \mathcal{B}Q \left(\sum_{L<l\le L_1} \sum_{1\le i\le Q} \sum_{R_{i-1}<k\le R_i} |a_k|^2 \right) \\ &\quad + \mathcal{B}Q \left(\sum_{1\le i\le Q} \sum_{\substack{R_{i-1}<k_1,k_2\le R_i \\ k_1\le k_2}} |a_{k_1}||a_{k_2}| \left| \sum_{L<l\le L_1} e((k_1^c - k_2^c) l^c x + (k_1^d - k_2^d) l^d y) \right| \right) \\ &\ll \mathcal{B}Q \left(\mathcal{A}L + \sum_{1\le h\le s} \sum_{1\le i\le Q} \sum_{R_{i-1}<k\le R_i-h} |a_k||a_{k+h}| \left| \sum_{L<l\le L_1} e(f(l)) \right| \right), \end{aligned} \tag{5.1}$$

where

$$f(l) = ((k+h)^c - k^c) l^c x + ((k+h)^d - k^d) l^d y.$$

For the exponential sum $\sum e(f(l))$ of (5.1), we consider

$$f^j(l) \asymp \left(((k+h)^c - k^c) L^{c-1} x + ((k+h)^d - k^d) L^{d-1} y \right) L^{1-j}.$$

An application of Lemma 3.2 with the exponent pair $(\frac{13}{84}, \frac{55}{84})$ yields

$$\begin{aligned} \left| \sum_{L<l\le L_1} e(f(l)) \right| &\ll \left(((k+h)^c - k^c) L^{c-1} x + ((k+h)^d - k^d) L^{d-1} y \right)^{13/84} L^{55/84} \\ &\ll \left(((k+h)^c - k^c) L^{c-1} K_1 + ((k+h)^d - k^d) L^{d-1} K_2 \right)^{13/84} L^{55/84} \\ &\ll s^{13/84} (R^{13(c-1)/84} L^{(13c+42)/84} K_1^{13/84} + R^{13(d-1)/84} L^{(13d+42)/84} K_2^{13/84}). \end{aligned}$$

Substituting this upper bound into (5.1) yields

$$\begin{aligned}|V|^2 \ll& \mathcal{B}Q\Big[\mathcal{A}L + s^{13/84}(R^{13(c-1)/84}L^{(13c+42)/84}K_1^{13/84}\\ &+R^{13(d-1)/84}L^{(13d+42)/84}K_2^{13/84})\sum\nolimits_1\Big],\end{aligned}$$

where

$$\sum\nolimits_1 = \sum_{1\le h\le s}\sum_{1\le i\le Q}\sum_{R_{i-1}<k\le R_i-h}|a_k||a_{k+h}|.$$

Then we need to handle $\sum_1$. Following from Cauchy's inequality, we obtain

$$\begin{aligned}\sum\nolimits_1 &\le \sum_{1\le h\le s}\sum_{X^{1/4}<k\le R-h}|a_k||a_{k+h}|\\ &\le \sum_{1\le h\le s}\left(\sum_{X^{1/4}<k\le R-h}|a_k|^2\right)^{1/2}\left(\sum_{X^{1/4}<k\le R-h}|a_{k+h}|^2\right)^{1/2}\\ &\ll s\mathcal{A}.\end{aligned}$$

From the above estimate and $Q \ll R/s$, we get

$$\begin{aligned}|V|^2 \ll \mathcal{AB}LR\Big[&1/s + s^{13/84}(R^{13(c-1)/84}L^{(13c-42)/84}K_1^{13/84}\\ &+R^{13(d-1)/84}L^{(13d-42)/84}K_2^{13/84})\Big].\end{aligned}\tag{5.2}$$

By taking $s^{-1} = s^{13/84}(R^{13(c-1)/84}L^{(13c-42)/84}K_1^{13/84} + R^{13(d-1)/84}L^{(13d-42)/84}K_2^{13/84})$, we can obtain the value of s, *i.e.*,

$$s = R^{13(1-c)/97}L^{(42-13c)/97}K_1^{-13/97} + R^{13(1-d)/97}L^{(42-13d)/97}K_2^{-13/97},$$

then $s \in [1, R]$. Substituting this value of s into (5.2) yields

$$|V|^2 \ll \mathcal{AB}X^{(26v+139)/194}\log^{-26} X.$$

Hence, the proof of this lemma has been completed. □

Lemma 5.3. *The definition of a_k, b_l is as above. Let*

$$V' = \sum_{L<l\le L_1}\sum_{X^{1/4}<k\le \frac{X}{l}} a_k b_l e((kl)^c x + (kl)^d y),$$

where $L < L_1 \le 2L, L \le X$.
For $|a'_k| \le |a_k|, |b'_l| \le |b_l|$, we get

$$|V'| \ll (\log X)\left|\sum_{L<l\le L_1}\sum_{X^{1/4}<k\le \frac{X}{L}} a'_k b'_l e((kl)^c x + (kl)^d y)\right|.$$

Proof. This lemma is Lemma 9 of Tolev [10]. □

Lemma 5.4. *Suppose that* $(x, y) \in D_2$. *Then*

$$|P(x, y)| \ll X^{(26v+333)/388} \log^{-9} X.$$

Proof. For the case $x \in [\tau_1, K_1], y \in [0, K_2]$.
Put

$$W_0(x, y) = \sum_{X^{1/4} < m \leq X} \Lambda(m) e(m^c x + m^d y),$$
$$W_1(x, y) = \sum_{X^{1/4} < m \leq \lambda X} \Lambda(m) e(m^c x + m^d y).$$

Obviously, we have

$$P(x, y) = W_0(x, y) - W_1(x, y) + O(X^{1/2}).$$

We only need to prove that

$$|W_0(x, y)|, |W_1(x, y)| \ll X^{(26v+333)/388} \log^{-9} X. \tag{5.3}$$

Next, we only need to estimate $W_0(x, y)$. The proof for the estimate of $W_1(x, y)$ is similar. Put

$$J_1 = \sum_{m \leq X^{1/4}} \mu(m) \sum_{r \leq \frac{X}{m}} (\log r) e((rm)^c x + (rm)^d y),$$
$$J_2 = \sum_{g \leq X^{1/2}} \sum_{n \leq \frac{X}{g}} c_g e((gn)^c x + (gn)^d y),$$
$$J_3 = \sum_{X^{1/4} < k \leq X^{3/4}} \sum_{X^{1/4} < l \leq \frac{X}{k}} a_k \Lambda(l) e((kl)^c x + (kl)^d y),$$

where $|a_k| \leq d(k)$ and $|c_g| \leq \log g$.

Using Vaughan's identity (Vaughan [11]), we obtain

$$W_0(x, y) = J_1 - J_2 - J_3. \tag{5.4}$$

For J_1, we get

$$\begin{aligned} |J_1| &\leq \sum_{m \leq X^{1/4}} \left| \sum_{r \leq \frac{X}{m}} (\log r) e((rm)^c x + (rm)^d y) \right| \\ &\ll \sum_{m \leq X^{1/4}} \left| (\log^2 X) \max_{L_2 \in [L, L_1]} \left| \sum_{L < r \leq L_2} e(h(r)) \right| \right|, \end{aligned} \tag{5.5}$$

where $L < L_1 \leq \min(2L, X/m)$ and $h(r) = (rm)^c x + (rm)^d y$. For the exponential sum $\sum e(h(r))$, we consider $h^j(r) \asymp (L^{c-1} m^c x + L^{d-1} m^d y) L^{1-j}$. Then following from Lemma 3.2 and choosing exponent pair $(\frac{13}{84}, \frac{55}{84})$, we obtain

$$\begin{aligned}\left|\sum_{L<r\le L_1} e(h(r))\right| &\ll (L^{c-1}m^cK_1+L^{d-1}m^dK_2)^{13/84}L^{55/84}\\ &\ll X^{13c/84}L^{1/2}K_1^{13/84}+X^{13d/84}L^{1/2}K_2^{13/84}.\end{aligned} \tag{5.6}$$

Substituting the estimate (5.6) into (5.5) yields

$$|J_1| \ll X^{(26v+105)/168}\log^{-11} X. \tag{5.7}$$

Put

$$\begin{aligned}J_{31} &= \sum_{X^{1/2}<l\le X^{3/4}}\ \sum_{X^{1/4}<k\le \frac{X}{l}} a_k\Lambda(l)e((kl)^cx+(kl)^dy),\\ J_{32} &= \sum_{X^{1/2}<l\le X^{3/4}}\ \sum_{X^{1/4}<k\le \frac{X}{l}} a_l\Lambda(k)e((kl)^cx+(kl)^dy),\\ J_{33} &= \sum_{X^{1/4}<l\le X^{1/2}}\ \sum_{X^{1/4}<k\le X^{1/2}} a_k\Lambda(l)e((kl)^cx+(kl)^dy).\end{aligned}$$

For J_3, we have

$$J_3 = J_{31}+J_{32}+J_{33}. \tag{5.8}$$

Let $|a'_k|\le|a_k|\le d(k)$ and $|b'_l|\le\Lambda(l)$. We split the summation range $X^{1/2}<l\le X^{3/4}$ and apply Lemma 5.3, which gives

$$\begin{aligned}|J_{31}| &\ll (\log^2 X)\left|\sum_{L<l\le L_1}\ \sum_{X^{1/4}<k\le \frac{X}{L}} a'_kb'_le((lk)^cx+(lk)^dy)\right|\\ &:= (\log^2 X)|J'_{31}(L)|.\end{aligned} \tag{5.9}$$

By the mean value estimates, we have

$$\sum_{k\le y} d^2(k)\ll y\log^3 y, \quad \sum_{l\le y}\Lambda^2(l)\ll y\log y.$$

Recall from Lemma 5.2, we have

$$\begin{aligned}|J'_{31}(L)| &\ll \left(\frac{X}{L}\log^3 X\cdot L\log X\right)^{1/2} X^{(26v+139)/388}\log^{-13} X\\ &= X^{(26v+333)/388}\log^{-11} X.\end{aligned}$$

Substituting this upper bound into (5.9) yields

$$|J_{31}| \ll X^{(26v+333)/388}\log^{-9} X. \tag{5.10}$$

Similarly, the sums $|J_{32}|$ and $|J_{33}|$ contribute $O(X^{(26v+333)/388}\log^{-9} X)$.

For J_2, it's sufficient to show that

$$\sum_{X^{1/4}<g\leq X^{1/2}}\sum_{n\leq X^{1/4}} c_g e((gn)^c x+(gn)^d y)\ll X^{3/4}\log X.$$

Write

$$\begin{aligned}
J_{21}&=\sum_{g\leq X^{1/4}}\sum_{n\leq \frac{X}{g}} c_g e((gn)^c x+(gn)^d y),\\
J_{22}&=\sum_{X^{1/4}<n\leq X^{1/2}}\ \sum_{X^{1/4}<g\leq X^{1/2}} c_g e((gn)^c x+(gn)^d y),\\
J_{23}&=\sum_{X^{1/2}<n\leq X^{3/4}}\ \sum_{X^{1/4}<g\leq \frac{X}{n}} c_g e((gn)^c x+(gn)^d y).
\end{aligned}$$

Then we can get

$$J_2=J_{21}+J_{22}+J_{23}+O(X^{3/4}\log X).$$

The estimation for J_{21} is similar to J_1:

$$|J_{21}|\ll X^{(26v+105)/168}\log^{-11}X. \tag{5.11}$$

Using the mean value estimate, we have $\sum_{g\leq y}\log^2 g\ll y\log^2 y$. The estimations for J_{22} and J_{23} are similar to J_{31}:

$$|J_{22}|,|J_{23}|\ll X^{(26v+333)/388}\log^{-9}X. \tag{5.12}$$

Obviously, we have

$$X^{(26v+105)/168}\log^{-11}X\leq X^{(26v+333)/388}\log^{-9}X. \tag{5.13}$$

Combining (5.4) and (5.7)-(5.13), we can obtain (5.3). □

Lemma 5.5. *For $P(x,y)$ defined by (2.3), we have*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|P(x,y)|^4\psi_{\varepsilon_1}(x)\psi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y\ll X^2\log^6 X.$$

Proof. This result is established as Lemma 14 in Tolev [6]. □

Proof of Proposition 5.1. By Lemma 5.4 and Lemma 5.5, we have

$$\begin{aligned}
|E_2|&\ll \max_{D_2}|P(x,y)|^{k-4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|P(x,y)|^4\psi_{\varepsilon_1}(x)\psi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y\\
&\ll \left(X^{(26v+333)/388}\log^{-9}X\right)^{k-4}X^2\log^6 X\\
&\ll X^{(26k^2+562k-1112)/(26k+672)}\log^{389}X\\
&=\frac{\varepsilon_1\varepsilon_2 X^{k-c-d}}{\log X}
\end{aligned}$$

which finishes the proof. □

6. Proof of the Theorem 1.1

Recalling the inequalities (2.2) and (2.4), we know

$$\begin{aligned} E &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P^k(x,y)e(-N_1x-N_2y)\psi_{\varepsilon_1}(x)\psi_{\varepsilon_2}(y)\mathrm{d}x\mathrm{d}y \\ &= E_1+E_2+E_3. \end{aligned}$$

We know from Proposition 4.1 and Proposition 5.1 that the bounds for E_1 and E_2:

$$|E_1| \gg \varepsilon_1\varepsilon_2X^{k-c-d}, |E_2| \ll \frac{\varepsilon_1\varepsilon_2X^{k-c-d}}{\log X}.$$

From Lemma 3.1, we can derive the upper bound

$$|E_3| \ll 1.$$

Then we obtain the bound for E as $E \gg \varepsilon_1\varepsilon_2X^{k-c-d}$.

Following from Lemma 3.3 and the bound of E, we get

$$G \gg \varepsilon_1\varepsilon_2X^{k-c-d},$$

which finishes the proof of Theorem 1.1.

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The authors report there are no conflicts of interest pertaining to this study.

Data availability statement

In the whole paper, there is no new data generation and analysis.

Author contribution statement

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